DOUBLE LIE ALGEBROIDS AND
SECOND-ORDER GEOMETRY, II [*]

K. C. H. Mackenzie
School of Mathematics and Statistics
University of Sheffield
Sheffield, S3 7RH
England
K.Mackenzie@sheffield.ac.uk

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Abstract

We complete the construction of the double Lie algebroid of a double Lie groupoid begun in the first paper of this title. We extend the construction of the tangent prolongation of an abstract Lie algebroid to show that the Lie algebroid structure of any $\mathcal{LA}$-groupoid may be prolonged to the Lie algebroid of its groupoid structure. In the case of a double groupoid, this prolonged structure for either $\mathcal{LA}$-groupoid is canonically isomorphic to the Lie algebroid structure associated with the other; this extends many canonical isomorphisms associated with iterated tangent and cotangent structures.

We calculate several examples from Poisson geometry. We show that the cotangent of any double Lie groupoid is a symplectic double groupoid and that the side groupoids of a symplectic double groupoid are Poisson groupoids in duality; thus the duals of the $\mathcal{LA}$-groupoids of any double groupoid are a pair of Poisson groupoids in duality.

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In the first paper of this title [13] we proposed a Lie theory for double Lie groupoids based on the known Lie theory of ordinary Lie groupoids, on various constructions in Poisson geometry, and on the special features of the differential geometry of the tangent bundle. In this paper we complete the lengthy construction of the double Lie algebroid of a double Lie groupoid which was begun in [13], and calculate the fundamental examples, concentrating on those which arise in Poisson geometry. We refer to the introduction of [13] for an overview of the programme and its ultimate aims. Here we describe the features specific to the present paper, by way of orientation.

Ordinary Lie algebroids may be viewed both as generalizations of Lie algebras—and therefore as vehicles for a generalized Lie theory [14]—and as abstractions of the tangent bundle of an ordinary manifold. (For the latter point of view see, for example, [22] or [16].) For double Lie algebroids, the model of the Lie theory of Lie groups and Lie algebras is a more distant one, and we will be chiefly concerned here with constructions which derive ultimately from the calculus possible on the double tangent bundle.

In [13] it was shown how a single application of the Lie functor to a double Lie groupoid S produces an $\mathcal{LA}$-groupoid, that is, a Lie groupoid object in the category of Lie algebroids. If one applies the Lie functor to, say, the vertical structure of S, then the $\mathcal{LA}$-groupoid is vertically a Lie algebroid and horizontally a Lie groupoid; this follows easily from the fact that the Lie functor preserves pullbacks and the diagrams which define a groupoid structure. One may then take the Lie algebroid of this horizontal groupoid and obtain a double vector bundle whose horizontal structure is a Lie algebroid. Interchanging the order of the processes yields a second double vector bundle with the Lie algebroid structure now placed vertically. The two double vector bundles may be identified by a map derived from the canonical involution in the double tangent bundle of S, and one thus obtains a double vector bundle all four sides of which have Lie algebroid structures; this is the double Lie algebroid of S. This is the construction as outlined in the introduction to [13]. It includes, amongst many others, the double tangent bundle of an arbitrary manifold, the Lie bialgebra of a Poisson group, and the double cotangent bundle of a Poisson manifold.

However, abstract $\mathcal{LA}$-groupoids frequently arise in nature without an underlying double Lie groupoid, most notably the cotangent $\mathcal{LA}$-groupoid of a Poisson groupoid or Poisson Lie group. We have therefore shown in §1 here that the Lie algebroid structure on an $\mathcal{LA}$-groupoid may be prolonged to the Lie algebroid of the groupoid structure. This extends, in particular, the construction of tangent Lie algebroid structures given by Ping Xu and the author in [15]. We must then prove (Theorem 2.3) that, in the case of the $\mathcal{LA}$-groupoids of a double Lie groupoid, the prolonged Lie algebroid structure on the double Lie algebroid of either $\mathcal{LA}$-groupoid coincides with the Lie algebroid of the Lie groupoid structure of the other. This result embodies many canonical isomorphisms known in special cases.

We believe that the basic simplicity of this construction, and the richness of the phenomena which it encompasses, establish beyond doubt that it is a natural and correct definition. On a future occasion we will consider the abstract notion of double Lie algebroid and its local integrability. In §3 to §5 of the present paper, we are concerned to demonstrate the range of examples covered by the notion of double Lie algebroid.

The paper concludes with a detailed examination of the example of symplectic and Poisson double groupoids, and the duality associated with the former. We give in §4 a double cotangent groupoid construction analogous to the cotangent of an ordinary groupoid [3], and show that this embodies the relationship between the groupoid structures and the symplectic

structure. The key to the cotangent double groupoid is the notion of core introduced in [2]. In §5 we give a simple proof that the side groupoids of a symplectic double groupoid are Poisson groupoids the Lie bialgebroids of which are dual, and that the core gives a symplectic realization of the double base. The double Lie algebroid is (for either structure) the cotangent Lie algebroid of the linearized Poisson structures associated to the dual Poisson groupoids; this is the object studied at length in [15] and elsewhere as embodying the properties of a Lie bialgebroid.

In §3 we calculate the double Lie algebroid associated with an affinoid structure: this is, as indicated by Weinstein [21], a pair of conjugate flat partial connections. We show that the infinitesimal version of the notion of butterfly diagram is equivalent (under mild conditions) to such a pair of connections.

We have kept to a minimum the repetition of material from [13] and so refer the reader there for basic definitions and notations and much else. We also make extensive use of the material in §5 and §7 of [15], and of [16]. One minor change from [13] is that the pullback of a vector bundle A over a map f is denoted $f^! A$ rather than $f^* A$, with other pullback notation modified accordingly.

1   THE DOUBLE LIE ALGEBROID OF AN $\mathcal{LA}$-GROUPOID

The construction of the double Lie algebroid of a double Lie groupoid will be made much easier if we begin by first considering the application of the Lie functor to an arbitrary $\mathcal{LA}$-groupoid as defined in [13, §4] (see Figure 1). Here Ω is both a Lie groupoid over base A

Figure 1:

(which is itself a Lie algebroid over M), and a Lie algebroid over G (which is a Lie groupoid over M); the two structures on Ω commute in the sense that the maps defining the groupoid structure are all Lie algebroid morphisms. It is further assumed that the double source map $(\widetilde{q}, \widetilde{\alpha}): \Omega \to G \underset{M}{\times} A$ is a surjective submersion.

Since Ω is a Lie groupoid over A we may take its Lie algebroid $A\Omega$; denote the bundle projecttion $A\Omega \to A$ by $\overset{\vee}{q}$. Since $\widetilde{q}\colon \Omega \to G$ is a morphism of groupoids, it induces a morphism of Lie algebroids $A(\widetilde{q})\colon A\Omega \to AG$. And since each of the other maps defining the vector bundle structure in $\Omega \to G$ (namely the addition, scalar multiplication, and zero section) are morphisms of Lie groupoids, we may apply the Lie functor to them, and we obtain a vector bundle structure on $A(\widetilde{q})\colon A\Omega \to AG$. As with the construction in [13, §4, p.216], of which this is a special case, a detailed proof is unnecessary: a vector bundle is defined by pullback diagrams, and the Lie functor preserves pullbacks and diagrams.

Proposition 1.1 *The above construction yields a double vector bundle*

$$
\begin{array}{ccc}
A\Omega & \xrightarrow{\ \overset{\vee}{q}\ } & A \\[2pt]
{\scriptstyle A(\widetilde{q})}\Big\downarrow & & \Big\downarrow{\scriptstyle q} \\[2pt]
AG & \xrightarrow[\ q_G\ ]{} & M.
\end{array}
\tag{1}
$$

PROOF. To verify the commutativity, it only needs to be recalled that a vector bundle, regarded as a Lie groupoid, is itself its own Lie algebroid, and, similarly, applying the Lie functor to a morphism of vector bundles leaves the morphism unchanged. ∎

Three sides of (1) have Lie algebroid structures, and we are now concerned to put such a structure on $A\Omega \to AG$ as well. This will take us until the end of the section. The method used is a generalization of the construction of a tangent Lie algebroid [15, 5.1].

The anchor is the composite of $A(\widetilde{a})\colon A\Omega \to ATG$ with the inverse of the canonical isomorphism $j_G\colon TAG \to ATG$ of [15, 7.1]; we denote it by $\mathbf{a}$. Using [15, 7.1], $\mathbf{a}$ is a morphism of Lie algebroids over $a\colon A \to TM$.

We define the bracket structure on $A\Omega \to AG$ in terms of sections of two specific types.

Definition 1.2 *A* star section *of the $\mathcal{LA}$-groupoid in Figure 1 is a pair (ξ, X) where $\xi \in \Gamma_G\Omega$, $X \in \Gamma A$, and we have $\widetilde{1} \circ X = \xi \circ 1$, $\widetilde{\alpha} \circ \xi = X \circ \alpha$.*

This is a weakened form of the notion of morphic section considered in [13, 4.3]. The terminology comes from [7, 3.2].

Lemma 1.3 *Given any $X \in \Gamma A$, there is a star section (ξ, X) of the $\mathcal{LA}$-groupoid in Figure 1.*

PROOF. Define $\eta \in \Gamma_G\Omega$ by setting $\eta(1_m) = \widetilde{1}_{X(m)}$ for $m \in M$, and extending over G. Then $\mu\colon G \to A$ defined by $\mu(g) = \widetilde{\alpha}(\eta(g)) - X(\alpha g)$ is a section of the pullback bundle $\alpha^! A$. Thanks to the double source condition, there is a $\zeta \in \Gamma_G\Omega$ with $\widetilde{\alpha} \circ \zeta = \mu$; we can also require that ζ vanish on all $1_m \in G$. Now $\xi = \eta - \zeta$ is a section of the required type. ∎

The conditions on a star section ensure that it is possible to apply the Lie functor to it; we obtain a linear section $A(\xi)\colon AG \to A\Omega$ of $A(\widetilde{q})$ for which $\overset{\vee}{q} \circ A(\xi) = X \circ q_G$. As in [13,

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4.3] it is clear that the bracket of two star sections is a star section. We now define

$$[A(\xi), A(\eta)] = A([\xi, \eta]) \tag{2}$$

for any two star sections (ξ, X), (η, Y).

To define sections of the second type, we must make a detour to determine the core structure of the double vector bundle (1) in terms of the core of the underlying $\mathcal{LA}$-groupoid. The core Lie algebroid of an $\mathcal{LA}$-groupoid was defined in [13, §5].

Proposition 1.4 *Let $(\Omega; G, A; M)$ be an $\mathcal{LA}$-groupoid with core Lie algebroid K. Denote by K_{vb} the vector bundle underlying K. Considering K_{vb} as a groupoid, let it act on $q_A \colon A \to M$ by $(\kappa, X) \mapsto \partial_A(\kappa) + X$, where $\kappa \in K, X \in A$, $q_K(\kappa) = q_A(X)$. Then*

$$K_{vb} \ltimes q_A \rightarrowtail \Omega \xrightarrow{\widetilde{q}} \!\!\!\!\!\gg G \tag{3}$$

is an exact sequence of Lie groupoids, where the injection is $(\kappa, X) \mapsto \kappa + \widetilde{1}_X$.

Let the Lie algebroid K act on $\beta \colon G \to M$ by sending $\kappa \in \Gamma K$ to $\widetilde{a}(\kappa) \in \Gamma TG$, the right-invariant vector field corresponding to $\widetilde{a}(\kappa) \in \Gamma AG$. Then

$$K \ltimes \beta \rightarrowtail \Omega \xrightarrow{\widetilde{\alpha}} \!\!\!\!\!\gg A \tag{4}$$

is an exact sequence of Lie algebroids, where the injection is $(\kappa, g) \mapsto \kappa \widetilde{0}_g$.

PROOF. Take $(\kappa_1, X_1), (\kappa_2, X_2) \in K \ltimes q_A$ with $X_1 = \partial_A(\kappa_2) + X_2$. Multiplying the images in Ω, and using the interchange law between the vector bundle and groupoid structures, we have

$$
\begin{aligned}
(\kappa_1 + \widetilde{1}_{X_1})(\kappa_2 + \widetilde{1}_{X_2}) &= (\kappa_1 + \widetilde{1}_{X_2} + \widetilde{1}_{\partial_A(\kappa_2)})(\widetilde{1}_{X_2} + \kappa_2) \\
&= (\kappa_1 + \widetilde{1}_{X_2})\widetilde{1}_{X_2} + \widetilde{1}_{\partial_A(\kappa_2)}\kappa_2 \\
&= \kappa_1 + \widetilde{1}_{X_2} + \kappa_2,
\end{aligned}
$$

which is the image of $(\kappa_1 + \kappa_2, X_2)$ under the injection. So the injection is a groupoid morphism. Exactness is easily verified.

The injection in the second statement preserves the anchors since

$$\widetilde{a}(\kappa \widetilde{0}_g^{\Omega}) = \widetilde{a}(\kappa)\widetilde{0}_g^{TG} = T(R_g)(\widetilde{a}(\kappa)) = \overrightarrow{\widetilde{a}(\kappa)}(g),$$

using the fact that $\widetilde{a}$ is a morphism of groupoids. It suffices to verify bracket preservation on sections of the form $1 \otimes \kappa \in \Gamma(K \ltimes \beta)$, where $\kappa \in \Gamma K$. This follows from (22) in [13, §5]. ∎

For the $\mathcal{LA}$-groupoid $(TG; G, TM; M)$ of a Lie groupoid G, these two sequences are

$$A_{vb}G \ltimes p_M \rightarrowtail TG \xrightarrow{p_G} \!\!\!\!\!\gg G \quad \text{and} \quad AG \ltimes \beta \rightarrowtail TG \xrightarrow{\mathcal{R}} TG \xrightarrow{T(\alpha)} \!\!\!\!\!\gg TM,$$

where $\mathcal{R}$ is the right translation map $(X, g) \mapsto T(R_g)(X)$.

For completeness, we state the corresponding results for double Lie groupoids.

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Proposition 1.5 *Let $(S; H, V; M)$ be a double Lie groupoid with core groupoid C (see [2] or [13, §2]). Then*

$$C \ltimes \beta_H \rightarrowtail S_V \xrightarrow{\widetilde{\alpha}_H} \hspace{-1em}\gg V \tag{5}$$

is an exact sequence of Lie groupoids, where C acts on β_H by $(k, h) \mapsto \partial_H(k)h$ and the injection is $(k, h) \mapsto k \,\square\, \widetilde{1}_h^V$.

Likewise

$$C \ltimes \beta_V \rightarrowtail S_H \xrightarrow{\widetilde{\alpha}_V} \hspace{-1em}\gg H \tag{6}$$

is an exact sequence of Lie groupoids, where C acts on β_V by $(k, v) \mapsto \partial_V(k)v$ and the injection is $(k, v) \mapsto k \,\boxminus\, \widetilde{1}_v^H$.

When S is a locally trivial double Lie groupoid, the exact sequences (7), (8) of [13, §2] follow easily from (5), (6).

Proposition 1.6 *Let $(S; H, V; M)$ be a double Lie groupoid with core groupoid C. Then AC, the Lie algebroid of C, identifies canonically with the core Lie algebroid of the $\mathcal{LA}$-groupoid $A_H S$ and with the core Lie algebroid of the $\mathcal{LA}$-groupoid $A_V S$.*

PROOF. Let K be the core of $A_H S$. Fix $m \in M$. Then a curve c_t in the α-fibre of C above m with $c_0 = 1_m^C$ is also a curve in the α-fibre of S_H above 1_m^V, with $c_0 = 1_m^2$, the double identity at m. Thus there is a vector bundle morphism $AC \to A_H S$, which is easily seen to be an isomorphism over M onto K. Now applying the Lie functor to (6) and comparing it with (4) shows that this is an isomorphism of Lie algebroids. ∎

Returning to the $\mathcal{LA}$-groupoid in Figure 1, we can now apply the Lie functor to (3), and obtain an exact sequence of Lie algebroids

$$K^0 \ltimes q_A \rightarrowtail A\Omega \xrightarrow{A(\widetilde{q})} \hspace{-1em}\gg AG. \tag{7}$$

Here K^0 is the vector bundle underlying K equipped with the zero anchor and zero bracket: it is the Lie algebroid of K_{vb}. The action of K^0 on $q_A \colon A \to M$ is $\kappa^\dagger = \partial_A(\kappa)^\uparrow$, where for any $X \in \Gamma A$ we denote by $X^\uparrow$ the vertical vector field on A defined by

$$X^\uparrow(F)(Y) = \frac{d}{dt} F(Y + tX(qY)) \Big|_0$$

where $F \in C(A)$ and $Y \in A$. In [15, (23)], $X^\uparrow$ is denoted $\check{X}$ and called the *core vector field*, but here we use the notation $X^\uparrow$; in the case $A = TM$, $X^\uparrow$ is the vertical lift of a vector field on M to a vector field on TM.

If we disregard the Lie algebroid structures in (7), we can rewrite it as an exact sequence of vector bundles over A:

$$q_A^! K \rightarrowtail A\Omega \xrightarrow{A(\widetilde{q})^!} \hspace{-1em}\gg q_A^! AG. \tag{8}$$

From general properties of vector bundles [13, Proposition 1.2], the next result then follows.

Proposition 1.7 *The core of the double vector bundle (1) is K_{vb}, the vector bundle underlying the core of Ω.*

We think of elements of $A\Omega$ as derivatives

$$\Xi = \frac{d}{dt}\xi_t\Big|_0$$

where ξ_t is a curve in a fixed α-fibre Ω_X of Ω with $\xi_0 = \widetilde{1}_X$. Thus $\overset{\vee}{q}(\Xi) = X$. If we write $X_t = \widetilde{\beta}(\xi_t)$ and $g_t = \widetilde{q}(\xi_t)$ then

$$\frac{d}{dt}X_t\Big|_0 = \overset{\vee}{a}(\Xi) \in TA \qquad \text{and} \qquad \frac{d}{dt}g_t\Big|_0 = A(\widetilde{q})(\Xi) \in AG.$$

If k is an element of the core K of Ω, then it has $\widetilde{\alpha}(k) = 0_m$, $\widetilde{q}(k) = 1_m$ and $\widetilde{\beta}(k) = Y$ for some $m \in M$ and $Y = \partial_A(k) \in A$. The corresponding element of $A\Omega$ is $\frac{d}{dt}(tk)\big|_0$, where tk is the scalar multiplication in the bundle $\Omega \to G$. We denote this by $\overline{k}$ if confusion is likely. Note that, using the notation of [15, §5], $\overset{\vee}{a}(\overline{k}) = \overline{Y}$, the core element of TA corresponding to $Y \in A$.

The injection in (7) can now be written in either of two ways:

$$(k, X) \mapsto \frac{d}{dt}(tk + \widetilde{1}_X)\Big|_0 = \overline{k} +\!\!+ \widetilde{0}_X$$

where $+\!\!+$ denotes the addition in $A\Omega \to AG$ and $\widetilde{0}_X$ is the zero in $A\Omega \to A$ above X. Given a section $\kappa \in \Gamma K$, we define a section $\kappa^\diamond$ of $A\Omega \to A$ by

$$\kappa^\diamond(X) = \overline{\kappa(q_A X)} +\!\!+ \widetilde{0}_X = \frac{d}{dt}(\widetilde{1}_X + t\kappa(q_A X))\Big|_0. \tag{9}$$

We call $\kappa^\diamond$ the *core section* of $A\Omega \to A$ corresponding to κ.

Proposition 1.8 *For $\kappa, \lambda \in \Gamma K$ and $f \in C(M)$ we have*

$$(\kappa + \lambda)^\diamond = \kappa^\diamond + \lambda^\diamond, \qquad (f\kappa)^\diamond = (f \circ q_A)\kappa^\diamond, \qquad [\kappa^\diamond, \lambda^\diamond] = 0, \qquad \overset{\vee}{a}(\kappa^\diamond) = \partial_A(\kappa)^\uparrow.$$

PROOF. The first two statements are trivial. For the third, note that $\kappa^\diamond$ is the image under the injection in (7) of the pullback section $1 \otimes \kappa$ of $q_A^! K$. In any action Lie algebroid one has $[1 \otimes \kappa, 1 \otimes \lambda] = 1 \otimes [\kappa, \lambda]$ [7, 2.4], and since the acting Lie algebroid is K^0, we have $[\kappa, \lambda] = 0$.

Since the injection in (7) is a Lie algebroid morphism, $\overset{\vee}{a}(\kappa^\diamond(X)) = a'(\kappa(m), X)$, where a' is the anchor for $K^0 \ltimes q_A$ and $m = q_A X$. But we know that the anchor of an action Lie algebroid is given by the action, and so $a'(\kappa(m), X) = \kappa^\dagger(X) = \partial_A(\kappa)^\uparrow(X)$. ∎

Proposition 1.9 (i). *The anchor $\overset{\vee}{a}: A\Omega \to TA$ is a morphism of double vector bundles over $a_G: AG \to TM$ and id_A with core morphism $\partial_A: K \to A$.*

(ii). *The map $\mathbf{a}: A\Omega \to TAG$ is a morphism of double vector bundles over $a: A \to TM$ and id_{AG} with core morphism $\partial_{AG}: K \to AG$.*

PROOF. Only the statements concerning the core morphisms require proof. For the first statement the proof is essentially the same as for the last result of Proposition 1.8. For the second, it suffices to prove that $A(\widetilde{a})\colon A\Omega \to ATG$ has core morphism ∂_{AG}. Now for $k \in K$,

$$A(\widetilde{a})\left(\left.\frac{d}{dt}(tk)\right|_0\right) = \left.\frac{d}{dt}t\widetilde{a}(k)\right|_0 = \left.\frac{d}{dt}t\partial_{AG}(k)\right|_0 = \overline{\partial_{AG}(k)}.$$

∎

From Proposition 1.7 it follows that there is an exact sequence of vector bundles over AG, a companion to (7),

$$q_G^!K \rightarrowtail A\Omega \xrightarrow{\overset{\vee}{q}{}^!} \twoheadrightarrow q_G^!A, \tag{10}$$

where the injection is $(k, x) \mapsto A(0)(x) + \overline{k}$.

Again, given a section κ of K, there is a section $\kappa^\heartsuit$ of $A\Omega \to AG$ defined by

$$\kappa^\heartsuit(x) = A(0)(x) + \overline{\kappa(q_{AG}(x))} \tag{11}$$

for $x \in AG$. We call $\kappa^\heartsuit$ the *core section* of $A\Omega \to AG$ corresponding to κ. The construction $\kappa \mapsto \kappa^\heartsuit$ generalizes the construction $X \mapsto \widehat{X}$ for a vector bundle in [15, §5].

Proposition 1.10 *For $\kappa, \lambda \in \Gamma K$ and $f \in C(M)$ we have*

$$(\kappa + \lambda)^\heartsuit = \kappa^\heartsuit + \lambda^\heartsuit, \qquad (f\kappa)^\heartsuit = (f \circ q_{AG})\kappa^\heartsuit, \qquad \mathbf{a} \circ \kappa^\heartsuit = \partial_{AG}(\kappa)^\uparrow.$$

PROOF. The first two statements are trivial, and the last follows from Proposition 1.9(ii). ∎

It is the core sections $\kappa^\heartsuit$, $\kappa \in \Gamma K$, that together with those of the form $A(\xi)$ for (ξ, x) a star section, will allow us to define the Lie algebroid structure on $A\Omega \to AG$.

Proposition 1.11 *The sections of the form $A(\xi)$, where (ξ, X) is a star section, and $\kappa^\heartsuit$, where $\kappa \in \Gamma K$, generate $\Gamma_{AG}A\Omega$.*

PROOF. Take $\Xi \in A\Omega$ with $A(\widetilde{q})(\Xi) = x$ and $\overset{\vee}{q}(\Xi) = X(m)$. Extend $X(m) \in A$ to a section X of A. By Lemma 1.3, there is a star section (ξ, X) of Ω. We now have $\overset{\vee}{q}(A(\xi)(x)) = X(m)$ and so, by (10), we have

$$\Xi = A(\xi)(x) + \kern-0.6em+\; \kappa^\heartsuit(x)$$

for some $\kappa \in \Gamma K$. ∎

For $\kappa, \lambda \in \Gamma K$ we define

$$[\kappa^\heartsuit, \lambda^\heartsuit] = 0. \tag{12}$$

It remains to define brackets of the form $[A(\xi), \kappa^\heartsuit]$. We do this by an extension of the methods developed in [16].

For $\kappa \in \Gamma K$, denote (as in [13, §5]) by $\overline{\kappa}$ the element of $\Gamma_G\Omega$ given by $\overline{\kappa}(g) = \kappa(\beta g)\widetilde{0}_g$. Then $\overline{\kappa} \overset{\alpha}{\sim} 0$. If (ξ, X) is a star section, we have $\xi \overset{\alpha}{\sim} X$ and so it follows that $[\xi, \overline{\kappa}] \overset{\alpha}{\sim} 0$. Define

$$D_\xi(\kappa) = [\xi, \overline{\kappa}] \circ 1 \in \Gamma K. \tag{13}$$

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Lemma 1.12 *Let (ξ, X) and (η, Y) be star sections of Ω and let κ be a section of K. Then:*

(i). *If $\zeta \in \Gamma_G\Omega$ is any section with $\zeta \circ 1 = \kappa$, we have $D_\xi(\kappa) = [\xi, \zeta] \circ 1$.*

(ii). *$D_{[\xi,\eta]}(\kappa) = D_\xi(D_\eta(\kappa)) - D_\eta(D_\xi(\kappa))$.*

(iii). *For any $f \in C(M)$, $D_\xi(f\kappa) = fD_\xi(\kappa) + a(X)(f)\kappa$.*

PROOF. (i) It only needs to be proved that if $\eta \in \Gamma_G\Omega$ has $\eta \circ 1 = 0$ then $[\xi, \eta] \circ 1 = 0$. But $\widetilde{1}: A \to \Omega$ is a Lie algebroid morphism over $1: M \to G$, and $X \overset{1}{\sim} \xi$, so it follows that if $0 \overset{1}{\sim} \eta$, then $0 = [X, 0] \overset{1}{\sim} [\xi, \eta]$, which is to say that $[\xi, \eta] \circ 1 = 0$. From this (ii) now follows immediately.

(iii) For each $m \in M$ we have

$$
\begin{aligned}
D_\xi(f\kappa)(m) &= [\xi, (f \circ \beta)\overline{\kappa}](1_m) \\
&= f(m)D_\xi(\kappa)(m) + \widetilde{a}(\xi)(1_m)(f \circ \beta)\kappa(m)
\end{aligned}
$$

and $\widetilde{a}(\xi(1_m)) = \widetilde{a}(\widetilde{1}_{X(m)}) = T(1)(a(X(m)))$ since $\widetilde{a}: \Omega \to TG$ is a groupoid morphism over $a: A \to TM$. ∎

Thus $D_\xi: \Gamma K \to \Gamma K$ is a covariant differential operator over $a(X)$.

Proposition 1.13 *Let (ξ, X) be a morphic section. Then $[\xi, \overline{\kappa}] = \overline{D_\xi(\kappa)}$ for all $\kappa \in \Gamma K$, and D_ξ is a derivation of the bracket structure on ΓK.*

PROOF. Recall from [13, §5, p.230], that a section ζ of $\overline{K}$, the kernel of $\widetilde{\alpha}: \Omega \to A$, is G-equivariant if $\zeta(gh) = \zeta(g)\widetilde{0}_h$ for all compatible g, h. A G-equivariant section is accordingly determined by its values on the identity elements of G. Given any section ζ of $\overline{K}$, let $\zeta * \widetilde{0}$ denote the section $(g, h) \mapsto (\zeta(g), \widetilde{0}_h)$ of $\Omega * \Omega \to G * G$. Then ζ is G-equivariant iff $\zeta * \widetilde{0} \overset{\kappa}{\sim} \zeta$.

Now if (ξ, X) is morphic, we have $\xi * \xi \overset{\kappa}{\sim} \xi$, and so $\overline{\kappa} * \widetilde{0} \overset{\kappa}{\sim} \overline{\kappa}$ implies that $[\xi * \xi, \overline{\kappa} * \widetilde{0}] \overset{\kappa}{\sim} [\xi, \overline{\kappa}]$. Hence $[\xi, \overline{\kappa}] * \widetilde{0} \overset{\kappa}{\sim} [\xi, \overline{\kappa}]$ and so $[\xi, \overline{\kappa}]$ is G-equivariant. Since $[\xi, \overline{\kappa}] \circ 1 = D_\xi(\kappa)$, we must have $[\xi, \overline{\kappa}] = \overline{D_\xi(\kappa)}$.

The last statement now follows immediately. ∎

Thanks to Lemma 1.12 we can define

$$[A(\xi), \kappa^\heartsuit] = D_\xi(\kappa)^\heartsuit \tag{14}$$

for any star section (ξ, X) and any $\kappa \in \Gamma K$. By virtue of Proposition 1.11, we can now define a bracket on $\Gamma_{AG}A\Omega$ by extending the conditions (2), (12) and (14) by the nonlinearity condition

$$[\Xi, F \centerdot \Upsilon] = F \centerdot [\Xi, \Upsilon] + \mathbf{a}(\Xi)(F) \centerdot \Upsilon \tag{15}$$

which must hold for all $\Xi, \Upsilon \in \Gamma_{AG}(A\Omega)$ and $F \in C(AG)$.

Theorem 1.14 *Let $(\Omega; G, A; M)$ be an $\mathcal{LA}$-groupoid. Then the bundle $A\Omega \to AG$ is a Lie algebroid with respect to the anchor $\mathbf{a} = j_G^{-1} \circ A(\widetilde{a}): A\Omega \to TAG$ and the bracket just defined.*

We call this structure the *prolonged Lie algebroid structure* induced from $\Omega \to G$.

For the proof, we first need a Lemma, which involves the calculus of star vector fields developed in [16].

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Lemma 1.15 *For (ξ, X) a star section of $\Omega \to G$ with $\widetilde{a}(\xi) = x$, and $\kappa \in \Gamma K$, we have*

$$\mathbf{a} \circ A(\xi) = \widetilde{x}, \qquad \mathbf{a} \circ \kappa^\heartsuit = \partial_{AG}(\kappa)^\uparrow, \qquad \partial_{AG}(D_\xi(\kappa)) = D_x(\partial_{AG}(\kappa)).$$

PROOF. Firstly, we have $\mathbf{a} \circ A(\xi) = j_G^{-1} \circ A(\widetilde{a} \circ \xi) = j_G^{-1} \circ A(x) = \widetilde{x}$, as stated. The second identity is proved similarly.

For the third equation, we can calculate the LHS by finding $\widetilde{a}([\xi, \overline{\kappa}](1_m))$, since we know that $D_\xi(\kappa)(m)$ is a core element of Ω, and ∂_{AG} is the core of $\widetilde{a} \colon \Omega \to TG$. Now $\widetilde{a}$ is the anchor of a Lie algebroid structure, so

$$\widetilde{a}([\xi, \overline{\kappa}](1_m)) = [x, \widetilde{a}(\overline{\kappa})](1_m).$$

Write $W = \partial_{AG}(\kappa) \in \Gamma AG$. Then $\widetilde{a}(\overline{\kappa}) = \overrightarrow{W}$, the right-invariant vector field on G corresponding to W. Finally, $[x, \overrightarrow{W}](1_m) = D_x(W)(m)$. ■

PROOF OF THEOREM 1.14. It follows from the Lemma that $\mathbf{a}([\Xi, \Upsilon]) = [\mathbf{a}(\Xi), \mathbf{a}(\Upsilon)]$ for sections of the form $A(\xi)$ or $\kappa^\heartsuit$. Likewise, the Jacobi identity is easily checked for sections of these forms.

We must check that (15) is consistent with (2), (12), and (14) in the cases where they overlap. To verify consistency between (14) and (15) for $\Xi = A(\xi)$, $F = f \circ q_{AG}$, $\Upsilon = \kappa^\heartsuit$, we have

$$
\begin{aligned}
[A(\xi), (f \circ q_{AG}) \centerdot \kappa^\heartsuit] &= [A(\xi), (f\kappa)^\heartsuit] \\
&= D_\xi(f\kappa)^\heartsuit \\
&= (f D_\xi(\kappa) + a(X)(f)\kappa)^\heartsuit \\
&= (f \circ q_{AG}) \centerdot D_\xi(\kappa)^\heartsuit +\!\!+ (a(X)(f) \circ q_{AG}) \centerdot \kappa^\heartsuit \\
&= (f \circ q_{AG}) \centerdot [A(\xi), \kappa^\heartsuit] +\!\!+ (a(X)(f) \circ q_{AG}) \centerdot \kappa^\heartsuit
\end{aligned}
$$

and it remains to prove that $a(X)(f) \circ q_{AG} = \mathbf{a}(A(\xi))(f \circ q_{AG})$. But $\mathbf{a}(A(\xi)) = \widetilde{W}$, where $W = \widetilde{a}(\xi)$ is a star vector field on G lying above $a(X)$, and so the result follows.

Now taking $\Xi = \lambda^\heartsuit$, $F = f \circ q_{AG}$, $\Upsilon = \kappa^\heartsuit$, it is easy to see that both sides of (15) are zero.

The only other relation requiring a check of consistency is

$$A((f \circ \alpha)\xi) = (f \circ q_{AG}) \centerdot A(\xi),$$

where ξ is a star section, and $f \in C(M)$, and the proof is very similar to the preceding cases.

Thus the bracket is consistently defined, and the extension to the general case of $\mathbf{a}([\Xi, \Upsilon]) = [\mathbf{a}(\Xi), \mathbf{a}(\Upsilon)]$ and the Jacobi identity now follow from the nonlinearity condition (15). ■

Definition 1.16 *The double vector bundle (1) together with the given Lie algebroid structures on $A \to M$, $AG \to M$ and $A\Omega \to A$ and the Lie algebroid structure on $A\Omega \to AG$ just defined, is the* double Lie algebroid of the $\mathcal{LA}$-groupoid $(\Omega : G, A, M)$.

To justify this definition, we will in a future paper define an abstract concept of double Lie algebroid and verify that (1) satisfies it. In Example 1.18 below we show how Theorem 1.14 includes the construction of the Lie bialgebra of a Poisson Lie group (for example [11]) and the construction of the Lie bialgebroid of a Poisson groupoid [15]. Further examples of different types are given in §2. We first need the most basic example.

Example 1.17 Let A be any Lie algebroid on base M, and give $\Omega = A \times A$ the $\mathcal{LA}$-groupoid structure $(A \times A; M \times M, A; M)$ described at the end of [13, 4.4]. Then the Lie algebroid structure on $A\Omega = TA$ with base $AG = TM$ defined above coincides with that defined on $TA \to TM$ in [15, 5.1].

Example 1.18 Any Poisson Lie group G gives rise to an $\mathcal{LA}$-groupoid $(T^*G; G, \mathfrak{g}^*; \{1\})$, as described in [13, 4.12]. More generally, any Poisson groupoid $G \rightrightarrows P$ gives rise to an $\mathcal{LA}$-groupoid $(T^*G; G, A^*G; P)$ with the structures described in [15, §8]. We now show that the prolonged Lie algebroid structure on $AT^*G \to AG$ admits a second natural description in terms of the calculus of [16].

Consider at first a Lie groupoid $G \rightrightarrows P$. Then $T^*G \rightrightarrows A^*G$ has a natural structure of symplectic groupoid and there is consequently an isomorphism $s: T^*A^*G \to AT^*G$ of Lie algebroids over A^*G, where T^*A^*G has the cotangent Lie algebroid structure for the Poisson structure on A^*G, and AT^*G is the Lie algebroid of $T^*G \rightrightarrows A^*G$ [3]. It was shown in [15, 7.3] that this isomorphism has an alternative description. Firstly, by taking the dual of $j_G: TAG \to ATG$ over AG, and identifying the dual of $ATG \to AG$ with $AT^*G \to AG$ via the induced pairing, we obtain a map $j'_G: AT^*G \to T^*AG$ which is an isomorphism of double vector bundles over AG and A^*G. Composing this with the canonical isomorphism $R: T^*A^*G \to T^*AG$ from [15, 5.5], we have two isomorphisms of double vector bundles, shown in Figure 2. By [15, 7.3], $R \circ s^{-1} = j'_G$. In Figure 2 we have omitted the base P and identity

Figure 2:

arrows for clarity.

Now suppose that $G \Longrightarrow P$ is a Poisson groupoid. Then $AT^*G \to AG$ has the prolonged Lie algebroid structure (1.14) from the cotangent structure on $T^*G \to G$. On the other hand, $T^*AG \to AG$ has the cotangent Lie algebroid structure from the Poisson structure on AG dual to the Lie algebroid structure on A^*G [20].

Theorem 1.19 ([16, §7]) *For any Poisson groupoid $G \Longrightarrow P$, $j'_G : AT^*G \to T^*AG$ is an isomorphism of Lie algebroids over AG with respect to the structures just described.*

Figure 2 can now be regarded as depicting isomorphisms between three double Lie algebroids. Firstly, $(AT^*G; AG, A^*G; P)$ is the double Lie algebroid of $(T^*G; G, A^*G; P)$ and has the structures of 1.16. Secondly, $T^*A^*G \to A^*G$ is the cotangent Lie algebroid of the Poisson structure on A^*G dual to the Lie algebroid structure of AG. The structure on $T^*A^*G \to AG$ is either transported from the cotangent structure of $T^*AG \to AG$ via R, or is transported from the prolonged structure on $AT^*G \to AG$ via s. The structures on T^*AG may be described similarly or obtained from the self-dual nature of Lie bialgebroids [15].

In the case of Poisson Lie groups these structures may also be defined in terms of the coadjoint actions and dressing transformations.

2 THE DOUBLE LIE ALGEBROID OF A DOUBLE LIE GROUPOID

Throughout this section we consider a double Lie groupoid $(S; H, V; M)$, as defined in [13, Definition 2.1]. Applying the Lie functor to the vertical groupoid structure, as in [13, §4], we obtain the vertical $\mathcal{L}\mathcal{A}$-groupoid $(A_V S; H, AV; M)$; see Figure 3. The construction of §1 now

Figure 3:

leads to a double vector bundle

$$
\begin{array}{ccc}
A^2 S & \xrightarrow{\ \overset{\vee}{q}_H\ } & AV \\
A(\widetilde{q}_V) \big\downarrow & & \big\downarrow q_V \\
AH & \xrightarrow{\ q_H\ } & M
\end{array}
\tag{16}
$$

where $A^2S = A(A_VS)$ has two Lie algebroid structures: over base AV the structure of the Lie algebroid of the Lie groupoid $A_VS \rightrightarrows AV$, and over AH the prolongation of the Lie algebroid structure on $A_VS \to H$.

Equally well, we can first take the horizontal $\mathcal{LA}$-groupoid $(A_HS; AH, V; M)$ as in Figure 4.

$$
\begin{array}{ccc}
A_HS & \xrightarrow{\ \widetilde{q}_H\ } & V \\[2pt]
A(\widetilde{\alpha}_V), A(\widetilde{\beta}_V) \ \Big\Downarrow & & \Big\Downarrow \\[2pt]
AH & \xrightarrow{\hspace{2cm}} & M,
\end{array}
$$

Figure 4:

Applying the construction of §1 to A_HS yields a double vector bundle $A_2S = A(A_HS)$

$$
\begin{array}{ccc}
A_2S & \xrightarrow{\ A(\widetilde{q}_H)\ } & AV \\[2pt]
\overset{\vee}{q}_V \Big\downarrow & & \Big\downarrow q_V \\[2pt]
AH & \xrightarrow[\ q_H\ ]{} & M
\end{array}
\tag{17}
$$

and two Lie algebroid structures on A_2S: that over base AH being the Lie algebroid of $A_HS \rightrightarrows AH$, and that over AV being the prolongation of $A_HS \to V$.

Before proceeding, we need to consider a basic example.

Example 2.1 Let G be a Lie groupoid on M, and let $S = G \times G$ have the double Lie groupoid structure of [13, 2.3] with $H = M \times M$ and $V = G$. Then, as in [13, 4.4], the vertical and horizontal $\mathcal{LA}$-groupoids are respectively

$$
\begin{array}{ccc}
AG \times AG & \Longrightarrow & AG \\[2pt]
\Big\downarrow & & \Big\downarrow \\[2pt]
M \times M & \Longrightarrow & M,
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
TG & \xrightarrow{\hspace{1.5cm}} & G \\[2pt]
\Big\Downarrow & & \Big\Downarrow \\[2pt]
TM & \xrightarrow{\hspace{1.5cm}} & M.
\end{array}
\tag{18}
$$

We thus obtain $A^2S = TAG$ and $A_2S = ATG$.

In [15, 7.1], it was shown that $j_G: TAG \to ATG$ is an isomorphism of double vector

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bundles from

$$
\begin{array}{ccccccc}
TAG & \xrightarrow{\ p_{AG}\ } & AG & & ATG & \xrightarrow{\ A(p_G)\ } & AG \\
\Big\downarrow{\scriptstyle T(q_G)} & & \Big\downarrow & \ \text{to}\ & \ {\scriptstyle q_{TG}}\ \Big\downarrow & & \Big\downarrow \\
TM & \longrightarrow & M & & TM & \longrightarrow & M
\end{array}
$$

preserving AG and TM; j_G is a restriction of the canonical involution $J_G \colon T^2 G \to T^2 G$ associated with the manifold G. Further, [15, 7.1] shows that j_G is an isomorphism of the Lie algebroid structures over TM: it sends the tangent Lie algebroid structure of $T(AG) \to TM$ (which by Example 1.17 is the prolongation of $AG \times AG \to M \times M$) to the Lie algebroid structure $A(TG) \to TM$ arising from the tangent groupoid $TG \rightrightarrows TM$.

An element $\Xi \in TAG$ can be written as $\frac{\partial^2 g}{\partial t \partial u}(0,0)$ where $g \colon \mathbb{R}^2 \to G$ is a smooth map such that if $m = \beta \circ g \colon \mathbb{R}^2 \to M$ then $\alpha(g(t,u)) = m(t,0)$ and $g(t,0) = 1_{m(t,0)}$ for all t, u. The notation is meant to indicate that we first differentiate with respect to u and obtain a curve in AG, whose derivative with respect to t is Ξ. Accordingly, $p_{AG}(\Xi) = \frac{\partial g}{\partial u}(0,0)$, $T(q_G)(\Xi) = \frac{\partial m}{\partial t}(0,0)$ and

$$
j_G\left(\frac{\partial^2 g}{\partial t \partial u}(0,0) \right) = \frac{\partial^2 g}{\partial u \partial t}(0,0).
$$

It is easily seen that j_G is also an isomorphism of the Lie algebroid structures over AG. The structure in $ATG \to AG$ is defined by

$$
[A(\xi), A(\eta)] = A([\xi, \eta]), \qquad [A(\xi), X^\heartsuit] = D_\xi(X)^\heartsuit, \qquad [X^\heartsuit, Y^\heartsuit] = 0,
$$

where ξ and η are star vector fields on G and $X, Y \in \Gamma AG$. Now from [16, §3] we have

$$
j_G^{-1} \circ A(\xi) = \widetilde{\xi}, \qquad j_G^{-1} \circ X^\heartsuit = X^\uparrow,
$$

and

$$
[\widetilde{\xi}, \widetilde{\eta}] = \widetilde{[\xi, \eta]}, \qquad [\widetilde{\xi}, X^\uparrow] = D_\xi(X)^\uparrow, \qquad [X^\uparrow, Y^\uparrow] = 0, \tag{19}
$$

from which the result follows.

Thus j_G preserves both the horizontal and the vertical Lie algebroid structures.

∎

We now show that a similar phenomenon holds for any double Lie groupoid $(S; H, V; M)$.

Theorem 2.2 *The canonical involution $J_S \colon T^2 S \to T^2 S$ of the manifold S restricts to an isomorphism of double vector bundles $j_S \colon A^2 S \to A_2 S$ which preserves AH and AV.*

PROOF. We regard elements of $T^2 S$ as second derivatives

$$
\Xi = \frac{\partial^2 s}{\partial t \partial u}(0,0)
$$

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where $s\colon \mathbb{R}^2 \to S$ is a smooth square of elements, and the notation means that s is first differentiated with respect to u, yielding a curve $\xi_t = \frac{\partial s}{\partial u}(t,0)$ in TS with $\frac{d}{dt}\xi_t\big|_0 = \Xi$. Thus

$$\frac{\partial s}{\partial u}(0,0) = p_{TS}(\Xi), \qquad \frac{\partial s}{\partial t}(0,0) = T(p_S)(\Xi),$$

and $J_S(\Xi) = \frac{\partial^2 s}{\partial u \partial t}(0,0)$.

If $\Xi \in A^2 S$ then s can be chosen to be of the form

$$h(t,u)$$

$$v(t,u) \quad \boxed{\ s(t,u)\ } \quad v(0,u)$$

$$h(t,0)$$

where $h\colon \mathbb{R}^2 \to H$, $v\colon \mathbb{R}^2 \to V$ are smooth and satisfy the source and target conditions implicit in the diagram, and where $s(t,0) = \widetilde{1}^V_{h(t,0)}$, $s(0,u) = \widetilde{1}^H_{v(0,u)}$. We now have $\xi_t = \frac{\partial s}{\partial u}(t,0) \in A_V S$ with

$$\widetilde{q}_V(\xi_t) = h(t,0), \quad A(\widetilde{\beta}_V)(\xi_t) = \frac{\partial v}{\partial u}(t,0), \quad A(\widetilde{\alpha}_V)(\xi_t) = \frac{\partial v}{\partial u}(0,0),$$

and $\xi_0 = A(\widetilde{1}^H)(\frac{\partial v}{\partial u}(0,0))$. Consequently

$$A(\widetilde{q}_V)(\Xi) = \frac{\partial h}{\partial t}(0,0), \qquad \overset{\vee}{q}_H(\Xi) = \frac{\partial v}{\partial u}(0,0).$$

Interchanging the roles of t and u, it is easily seen that $j_S(\Xi) \in A_2 S$. Since j_S is a restriction of J_S it is injective and, by a dimension count, an isomorphism of double vector bundles. $\blacksquare$

Theorem 2.2 can also be proved by the diagram chasing methods used in [13, Proposition 1.5] and [15, 7.1].

Theorem 2.3 *Let $(S; H, V; M)$ be a double Lie groupoid. Then j_S, regarded as a morphism of vector bundles over AH, is an isomorphism of Lie algebroids from the prolonged structure on $A(A_V S) \to AH$ to the Lie algebroid of $A_H S \rightrightarrows AH$.*

PROOF. That j_S commutes with the anchors is proved by the same methods as in Theorem 2.2. It must be shown that for $\Xi \in A^2 S$, we have $j_H \circ \overset{\vee}{a}_V \circ j_S(\Xi) = A(\widetilde{a}_V)(\Xi)$. Taking $\Xi = \frac{\partial^2 s}{\partial t \partial u}(0,0)$ as above, both sides are equal to $\frac{\partial^2 h}{\partial t \partial u}(0,0)$.

The remainder of the proof depends on a series of subsidiary results, 2.4 to 2.8.

Given a star section (ξ, X) of $A_V S$, write $\widetilde{\xi} = j_S \circ A(\xi)$. As with sections of the Lie algebroid of any Lie groupoid, ξ induces a vector field $\overrightarrow{\xi}$ on S, which is right-invariant with respect to the groupoid structure on base H. Similarly, X induces a right invariant vector field $\overrightarrow{X}$ on V. Since (ξ, X) is a star section, it follows that $(\overrightarrow{\xi}, \overrightarrow{X})$ is a star vector field on the groupoid $S \rightrightarrows V$. Hence, by the constructions of [16], it induces a linear vector field $((\overrightarrow{\xi})^{\widetilde{\ }}, \overrightarrow{X})$ on $A_H S$, here regarded as a Lie algebroid over V.

On the other hand, $\widetilde{\xi}$ is a section of the Lie algebroid of $A_H S \rightrightarrows AH$, and therefore induces a right invariant vector field $(\widetilde{\xi})^{\overrightarrow{\ }}$ on $A_H S$.

Lemma 2.4 *For any star section (ξ, X) of the $\mathcal{LA}$-groupoid $A_V S$, $(\widetilde{\xi})^{\rightarrow} = (\overrightarrow{\xi})^{\sim}$.*

PROOF. Let $\operatorname{Exp} t\xi$ be the exponential for $\xi \in \Gamma(A_V S)$. We follow the conventions of [12], and assume for convenience that $\operatorname{Exp} t\xi$ is global. Similarly, let $\operatorname{Exp} tX$ be the exponential for $X \in \Gamma AV$. Since $\widetilde{\alpha}_H \circ \xi = X \circ \alpha_H$ and $\xi \circ 1^H = \widetilde{1}^H \circ X$, it follows that

$$\widetilde{\alpha}_H \circ \operatorname{Exp} t\xi = \operatorname{Exp} tX \circ \alpha_H, \qquad (\operatorname{Exp} t\xi) \circ 1^H = \widetilde{1}^H \circ \operatorname{Exp} tX.$$

The corresponding flow for $\overrightarrow{\xi}$ is $L_{\operatorname{Exp} t\xi} \colon S \to S$, $s \mapsto \operatorname{Exp} t\xi(\widetilde{\beta}_V s) \boxminus s$ and is a star map over $L_{\operatorname{Exp} tX} \colon V \to V$; that is;

$$\widetilde{\alpha}_H \circ L_{\operatorname{Exp} t\xi} = L_{\operatorname{Exp} tX} \circ \alpha_H \qquad \text{and} \qquad L_{\operatorname{Exp} t\xi} \circ \widetilde{1}^H = \widetilde{1}^H \circ L_{\operatorname{Exp} tX}. \tag{20}$$

We can therefore apply the Lie functor and get $A(L_{\operatorname{Exp} t\xi}) \colon A_H S \to A_H S$, which is a linear map over $L_{\operatorname{Exp} tX} \colon V \to V$. By [16, §3], this is the flow of $(\overrightarrow{\xi})^{\sim}$.

Now it is straightforward to check that $A(L_{\operatorname{Exp} t\xi})$ is also the flow of $(\widetilde{\xi})^{\rightarrow}$, and from this the result follows. ∎

Corollary 2.5 *For any star sections (ξ, X) and (η, Y) of the $\mathcal{LA}$-groupoid $A_V S$, $[\widetilde{\xi}, \widetilde{\eta}] = \widetilde{[\xi, \eta]}$.*

PROOF. As for any Lie groupoid, $[\widetilde{\xi}, \widetilde{\eta}]^{\rightarrow} = [(\widetilde{\xi})^{\rightarrow}, (\widetilde{\eta})^{\rightarrow}] = [(\overrightarrow{\xi})^{\sim}, (\overrightarrow{\eta})^{\sim}]$, by 2.4. Now this is $[\overrightarrow{\xi}, \overrightarrow{\eta}]^{\sim}$ by (19), and using $[\overrightarrow{\xi}, \overrightarrow{\eta}] = [\xi, \eta]^{\rightarrow}$ and 2.4 again, the result is clear. ∎

Next consider $\kappa \in \Gamma K$, where K is the core of $A_V S$. From (11) and (9), we have $\kappa^{\heartsuit}$, a section of $A^2 S \to AH$ and $\kappa^{\diamondsuit}$, a section of $A^2 S \to AV$. On the other hand, we can also consider K as the core of $A_H S$, and we denote the corresponding sections respectively by $\kappa^{\clubsuit}$, a section of $A_2 S \to AV$, and $\kappa^{\spadesuit}$, a section of $A_2 S \to AH$. This device depends on the fact that the cores of $A_H S$ and $A_V S$ can be canonically identified (Proposition 1.6). It is now easily checked that

$$j_S \circ \kappa^{\heartsuit} = \kappa^{\spadesuit}, \qquad j_S \circ \kappa^{\diamondsuit} = \kappa^{\clubsuit}. \tag{21}$$

In the next result, $\overline{\kappa}^H$ is the section of $A_H S \to V$ induced by κ. As a section of the vector bundle $A_H S$, it induces a vertical vector field on $A_H S$. On the other hand, $\kappa^{\spadesuit}$ is a section of the Lie algebroid of the groupoid $A_H S \rightrightarrows AH$, and consequently induces a right invariant vector field on $A_H S$.

Lemma 2.6 *For any $\kappa \in \Gamma K$, $(\overline{\kappa}^H)^{\uparrow} = (\kappa^{\spadesuit})^{\rightarrow}$.*

PROOF. Take any $\zeta \in A_H S|_v$ with $A(\widetilde{\beta}_V)(\zeta) = y \in A_n H$. We know from (9) that

$$\kappa^{\spadesuit}(y) = \frac{d}{dt}\left(A(\widetilde{1}^V)(y) + t\kappa(n)\right)\Big|_0$$

where $+$ is the addition in $A_H S \to V$. Right translating this curve by ζ, we get

$$(A(\widetilde{1}^V)(y) + t\kappa(n))\zeta = (A(\widetilde{1}^V)(y) + t\kappa(n))(\zeta + \widetilde{0}_v) = A(\widetilde{1}^V)(y)\zeta + t(\kappa(n)\widetilde{0}_v)$$

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by the interchange laws. This in turn is equal to $\zeta + t\overline{\kappa}^H(v)$, and

$$(\overline{\kappa}^H)^\uparrow(\zeta) = \frac{d}{dt}(\zeta + t\overline{\kappa}^H(v))\Big|_0$$

is just the definition of the vertical lift. ■

In the next result, the D operator on the RHS refers to the $\mathcal{LA}$-groupoid $A_V S$ and is as defined in (13), whereas the D on the LHS refers to the groupoid $S \Longrightarrow V$, and is defined in [16, §3].

Proposition 2.7 *Let (ξ, X) be a star section of $A_V S$, and let $\kappa \in \Gamma K$. Then*

$$D_{\overrightarrow{\xi}}(\overline{\kappa}^H) = \overline{D_\xi(\kappa)}^H.$$

PROOF. We must prove that for all $v \in V$,

$$[\overrightarrow{\xi}, \overrightarrow{\kappa}^H](\widetilde{1}_v^H) = [\xi, \overline{\kappa}^V](1_{\beta v}^H)\widetilde{0}_v.$$

On the LHS, $\overrightarrow{\xi}$ is the right invariant vector field on $S \Longrightarrow H$ corresponding to ξ, here considered as a star vector field on $S \Longrightarrow V$, whilst $\overrightarrow{\kappa}^H$ is the right invariant vector field, with respect to the horizontal groupoid structure on S, corresponding to $\overline{\kappa}^H \in \Gamma_V(A_H S)$. On the RHS we have the bracket of two sections of the Lie algebroid $A_V S$, and the multiplication in $A_H S \Longrightarrow AH$.

Assume as in the proof of Lemma 2.4 that $\mathrm{Exp}\, t\xi$ is a global exponential for ξ. Write $m = \beta v$ and $m_t = \beta \,\mathrm{Exp}\, tX(m)$. Then, using (20),

$$[\overrightarrow{\xi}, \overrightarrow{\kappa}^H](\widetilde{1}_v^H) = -\frac{d}{dt}(L_{\mathrm{Exp}\, t\xi})_*(\overrightarrow{\kappa}^H)(\widetilde{1}_v^H)\Big|_0 = -\frac{d}{dt}T(L_{\mathrm{Exp}\, t\xi})(\kappa(m_{-t})\widetilde{0}_{\mathrm{Exp}\, -tX(m)v})\Big|_0.$$

On the other hand, $[\xi, \overline{\kappa}^V](1_{\beta v}^H) = -\frac{d}{dt}T(I_{\mathrm{Exp}\, t\xi})(\kappa(m_{-t}))\Big|_0$ where I is the conjugation in $S \Longrightarrow H$;

$$I_{\mathrm{Exp}\, t\xi}(s) = \mathrm{Exp}\, t\xi(\widetilde{\beta}_V(s))\boxminus s \boxminus (\mathrm{Exp}\, t\xi(\widetilde{\alpha}_V(s)))^{-V}.$$

If we now write $\kappa(m_{-t}) = \frac{\partial}{\partial s}c(s,t)\Big|_{s=0}$ where $c(s,t)$ is a curve in C in the α-fiber above m_{-t}, then

$$T(I_{\mathrm{Exp}\, t\xi})(\kappa(m_{-t})) = \frac{\partial}{\partial s}\mathrm{Exp}\, t\xi(\widetilde{\beta}_V(c(s,t)))\boxminus c(s,t)\boxminus \widetilde{1}^H_{\mathrm{Exp}\, tX(m_{-t})^{-1}}\Big|_{s=0}.$$

Recalling that $(\mathrm{Exp}\, tX(m_{-t}))^{-1} = \mathrm{Exp}\, -tX(m)$, it now follows that

$$T(L_{\mathrm{Exp}\, t\xi})(\kappa(m_{-t})\widetilde{0}_{\mathrm{Exp}\, -tX(m)v}) = T(I_{\mathrm{Exp}\, t\xi})(\kappa(m_{-t}))\widetilde{0}_v,$$

and this completes the proof. ■

Corollary 2.8 *Let (ξ, X) be a star section of $A_V S$, and let $\kappa \in \Gamma K$. Then, in the Lie algebroid of $A_H S \Longrightarrow AH$,*

$$[\widetilde{\xi}, \kappa^\spadesuit] = D_\xi(\kappa)^\spadesuit.$$

PROOF. It suffices to prove that the corresponding right invariant vector fields on $A_H S$ are equal. Now, using Lemmas 2.4 and 2.6, the right invariant vector field corresponding to the LHS is $[(\overrightarrow{\xi})^{\widetilde{}}, (\overline{\kappa}^H)^{\uparrow}]$. Here $(\overrightarrow{\xi}, \overrightarrow{X})$ is a star vector field on $S \Longrightarrow V$, and so induces a vector field on $A_H S$. Similarly, $(\overline{\kappa}^H)^{\uparrow}$ is the vertical lift to $A_H S$ of a section of the Lie algebroid $A_H S$. Applying (19), we therefore get $D_{\overrightarrow{\xi}}(\overline{\kappa}^H)^{\uparrow}$. The result now follows from using Lemma 2.6 again. ■

We can now complete the proof of Theorem 2.3. By Proposition 1.11 we know that $\Gamma_{AH}(A^2 S)$ is generated by sections of the form $A(\xi)$, where (ξ, X) is a star section of $A_V S$, and those of the form $\kappa^{\heartsuit}$, where $\kappa \in \Gamma K$. Since we have $j_S \circ A(\xi) = \widetilde{\xi}$ and $j_S \circ \kappa^{\heartsuit} = \kappa^{\spadesuit}$, the equations from 2.5, 2.8 and 1.8 prove that j_S preserves the brackets. ■

Theorem 2.3 clearly includes the isomorphisms of Example 2.1. The isomorphisms of Example 1.18, however, are of a different type.

It is now clear that there are two, canonically isomorphic, double Lie algebroids associated with a double Lie groupoid. We make an arbitrary choice.

Definition 2.9 *The double vector bundle $A^2 S$ of (16), equipped with the Lie algebroid structure on $A^2 S \to AV$ of the Lie groupoid $A_V S \Longrightarrow AV$ and the prolonged Lie algebroid structure on $A^2 S \to AH$ from $A_V S \to H$, is the* double Lie algebroid *of the double Lie groupoid $(S; H, V; M)$.*

Example 2.10 Take $G = M \times M$ in Example 2.1, where M is any manifold. Thus $S = M^4$ consists of quadruples of points from M. From 2.1 we obtain $A^2 S = T^2 M$ and $A_2 S = T^2 M$ with $j_S = J_M$ the canonical involution of M. The prolonged structure on $T(p) : T^2 M \to TM$ has anchor $J = J_M$ and bracket $[\xi, \eta] = J[J\xi, J\eta]$ for all sections ξ, η of $T(p)$.

Example 2.11 Let H and V be Lie groupoids on the same base M, and let S be the double Lie groupoid $\square(H, V)$ of [13, 2.4]; assume that the anchors of H and V are suitably transversal.

Then, recalling [13, 4.5], the vertical $\mathcal{LA}$-groupoid $A_V S$ is

$$
\begin{array}{ccc}
\chi_H^{!!}(AV \times AV) \approx a_V^{!!}(TH) & \Longrightarrow & AV \\
\downarrow & & \downarrow \\
H & \Longrightarrow & M.
\end{array}
\tag{22}
$$

Here $\chi_H^{!!}(AV \times AV)$ is the pullback Lie algebroid of the cartesian square $AV \times AV$ across the anchor $\chi_H : H \to M \times M$, and $a_V^{!!}(TH)$ is the pullback Lie groupoid of $TH \Longrightarrow TM$ across $a_V : AV \to TM$.

Hence the horizontal structure on $A^2 S = A(A_V S)$ is the pullback Lie algebroid $a_V^{!!}(ATH)$ of $ATH \to TM$ across a_V. As a manifold

$$
A^2 S = TAV \underset{T^2 M}{\times} ATH,
$$

the pullback of $T(a_V)$ and a_{TH}. The anchor $\overset{\vee}{a}_H \colon A^2 S \to TAV$ is the natural map $TAV \underset{T^2M}{\times} ATH \to TAV$ and the bracket is of the standard form for pullbacks: see [7, §1].

Performing the horizontal and vertical differentiations in reverse order, we have

$$A_2 S = ATV \underset{T^2M}{\times} TAH,$$

the pullback of a_{TV} and $T(a_H)$. Over base AH the Lie algebroid structure on $A_2 S$ is the pullback of $ATV \to TM$ over $a_H \colon AH \to TM$. The canonical map $j_S \colon A^2 S \to A_2 S$ is $j_V \underset{T^2M}{\times} j_H^{-1}$. Theorem 2.3 now shows that the prolonged structure on $TAV \underset{T^2M}{\times} ATH \to AH$ is isomorphic to the pullback of $ATV \to TM$ across a_H.

Example 2.12 Let H and V be Lie groupoids on base M, and let $\phi \colon H \to V$ be a base-preserving morphism. Let $\Theta = \Theta(H, \phi, V)$ be the comma double groupoid of [13, 2.5]. Then, recalling [13, 4.7], the vertical $\mathcal{LA}$-groupoid $A_V \Theta$ is

$$
\begin{array}{ccc}
\alpha_H^{!!}(AV) \approx TH \ltimes a_V & \Longrightarrow & AV \\
\Big\downarrow & & \Big\downarrow \\
H & \Longrightarrow & M.
\end{array}
\tag{23}
$$

Here $\alpha_H^{!!}(AV)$ is the pullback Lie algebroid of AV across $\alpha_H \colon H \to M$, and $TH \ltimes a_V$ is the action groupoid arising from the action of $TH \rightrightarrows TM$ on a_V described in [13, 4.7].

To describe the induced infinitesimal action of ATH on $a_V \colon AV \to TM$, recall from [16] that in any Lie algebroid $A \to M$, a section X of A induces the vertical lift $X^{\uparrow}$ on A and also a linear vector field $(\widetilde{X}, a(X))$ characterized by $\widetilde{X}(\ell_\psi) = \ell_{L_X(\psi)}$ for all sections ψ of A^*.

The sections of $ATH \to TM$ are generated by those of the form $j_G \circ T(X)$ and those of the form $j_G \circ \widehat{X}$, where $X \in \Gamma AH$ (see [15, 7.1]). Now it is easy to check that the infinitesimal action is characterized by the equations

$$(j_G \circ T(X))^\dagger = \widetilde{\phi(X)}, \qquad (j_G \circ \widehat{X})^\dagger = \phi(X)^\uparrow. \tag{24}$$

Since the Lie algebroid of an action Lie groupoid is the corresponding action Lie algebroid, it follows that $A^2 \Theta \to AV$ is the action Lie algebroid $ATH \ltimes a_V$. As a manifold, $A^2 \Theta$ is $ATH \underset{TM}{\times} AV$, the pullback of q_{TH} and a_V.

On the other hand the horizontal $\mathcal{LA}$-groupoid of Θ is

$$
\begin{array}{ccc}
q_H^{!!}(V) \approx (AH \times AH) \ltimes \chi_V & \longrightarrow & V \\
\Big\Updownarrow & & \Big\Updownarrow \\
AH & \longrightarrow & M.
\end{array}
\tag{25}
$$

Here $q_H^{!!}(V)$ is the pullback groupoid of V over $q_H \colon AH \to M$ and $(AH \times AH) \ltimes \chi_V$ is the action Lie algebroid arising from the action of $AH \times AH$ on χ_V given in [13, 4.7].

Accordingly, $A_2\Theta \to AH$ is the pullback Lie algebroid $q_H^{!!}(AV)$. As a manifold, $A_2\Theta$ is $TAH \underset{TM}{\times} AV$, and j_Θ is $j_H^{-1} \underset{TM}{\times}$ id.

■

Notice throughout Examples 2.1, 2.11 and 2.12, that if the horizontal (say) structure is a pair groupoid, or a pullback, or an action groupoid, then the corresponding property is inherited by the double Lie algebroid. This follows from a version of [13, Lemma 4.6] formulated for $\mathcal{LA}$-groupoids and their double Lie algebroids.

Example 2.13 (Compare [20, §4.5].) Let $\Gamma \rightrightarrows P$ be any symplectic groupoid, and consider the double groupoid $S = \Gamma \times \Gamma$ of Example 2.1. Then the $\mathcal{LA}$-groupoids (18) can be given as

$$
\begin{array}{ccc}
T^*P \times T^*P & \Longrightarrow & T^*P \\
\downarrow & & \downarrow \\
P \times P & \Longrightarrow & P
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
T\Gamma & \longrightarrow & \Gamma \\
\| & & \| \\
TP & \longrightarrow & P
\end{array}
$$

where the canonical isomorphism of Lie algebroids $T^*P \to A\Gamma$ of [3] induces an isomorphism of $\mathcal{LA}$-groupoids. Now $T\Gamma \rightrightarrows TP$ is itself a symplectic groupoid with respect to the tangent symplectic structure on $T\Gamma$ and the tangent Poisson structure on TP and so $AT\Gamma \cong T^*(TP)$. Thus the double Lie algebroids are

$$
\begin{array}{ccc}
T(T^*P) & \longrightarrow & T^*P \\
\downarrow & & \downarrow \\
TP & \longrightarrow & P
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
T^*(TP) & \longrightarrow & T^*P \\
\downarrow & & \downarrow \\
TP & \longrightarrow & P
\end{array}
\qquad (26)
$$

with j_S now the canonical isomorphism $T(T^*P) \to T^*(TP)$. Theorem 2.3 applied to the vertical structures now shows that the prolongation of the cotangent Lie algebroid structure on T^*P is isomorphic to the cotangent Lie algebroid of the tangent Poisson structure on TP.

Both double vector bundles in (26) are duals of the double tangent bundle T^2P in the sense of Pradines' notion of dual (see [15, §5]). In these terms, the relations between the Lie algebroid structures in (26) reflect the fact that the canonical involution $J_P : T^2P \to T^2P$ is a Poisson automorphism with respect to the tangent of the tangent of the Poisson structure on P (see [4, 5.2]).

The double Lie algebroid $T(T^*P) \cong T^*(TP)$ and these isomorphisms can be defined for any (not necessarily integrable) Poisson manifold.

■

Several further examples requiring more extensive developments, such as the infinitesimal form of the theory associated with split double Lie groupoids and crossed modules [2] [13,

§2], will be treated in other papers. The case of the Lie algebroid structures associated with a vacant double Lie groupoid [13, §2] is dealt with by Mokri [17]. In the final two sections of the present paper we deal with the case of symplectic and Poisson double groupoids. In the next section we consider the infinitesimal theory associated with affinoids. §4 and §5 are independent of §3.

3 INFINITESIMAL STRUCTURES ASSOCIATED WITH AFFINOIDS

The notion of affinoid structure takes several different forms: as a ternary relation, and in the context of dual pairs in symplectic geometry, it was introduced by Weinstein [21]; as a generalization of principal bundles it was introduced by Kock [9] under the name of *pregroupoid*; as a form of Morita equivalence for groupoids it was introduced by Pradines [18] as a *butterfly diagram*. Forms of the notion, however, go back to the early part of the century; see the references in [21] and [9].

In [13, §3] we gave proofs of the equivalences between affinoid structures, butterfly diagrams and generalized principal bundles, using simple functorial constructions from groupoid and double groupoid theory. The key was to regard an affinoid structure as a type of double groupoid, called *principal* in [13, 3.2]. The interest of affinoid structures for us here is that the other two equivalent formulations, which are not overtly double structures, provide a means for testing the correctness of the double Lie algebroid construction. In this section we accordingly calculate the infinitesimal invariants associated with affinoid structures, butterfly diagrams and generalized principal bundles. Although the term "affinoid structure" should strictly refer to the ternary relation, in this section we will use it to mean the corresponding principal double Lie groupoid.

We first show that the double Lie algebroid of an affinoid structure consists of a pair of conjugate flat partial connections adapted to the two foliations of the affinoid (compare [21, 3.2]). Providing the leaves of both foliations are simply–connected, such a pair of partial connections is equivalent to the infinitesimal version of butterfly diagram (3.10, 3.12). However extending the Atiyah sequence construction to generalized principal bundles loses information and appears to be only of limited interest (3.13).

Some of the results of this section (3.9, 3.11) are special cases of results of Mokri [17]; this reflects the fact that an affinoid structure is a vacant double Lie groupoid. Here we can give more direct proofs which are also of interest, however.

Consider an affinoid structure S on a manifold M with surjective submersions $c\colon M \to Q_H$ and $b\colon M \to Q_V$. Orient the side groupoids as $H = R(b)$ and $V = R(c)$ with quotient groupoids respectively $G_h \rightrightarrows Q_H$ and $G_v \rightrightarrows Q_V$. In the notation of [13, §3], $G_h = \tau_1^V(S)$ and $G_v = \tau_1^H(S)$. Denote the fibrations defining the quotients by $\tilde{c}\colon R(b) \to G_h$ (over c) and $\tilde{b}\colon R(c) \to G_v$ (over b).

The morphism $\tilde{b}\colon R(c) \to G_v$ defines a vertical subbundle $T^{\tilde{b}}(R(c)) \subseteq T(R(c)) = R(T(c)) = TM \underset{TQ_H}{\times} TM$. On the other hand, applying the Lie functor to the morphism $\tilde{c}\colon R(b) \to G_h$ yields a Lie algebroid morphism $A(\tilde{c})\colon T^b M \to AG_h$.

Lemma 3.1 $T^{\tilde{b}}(R(c)) = R(A(\tilde{c}))$.

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PROOF. Take $(Z, X) \in T^{\widetilde{b}}(R(c))$. Thus $Z \in T_z(M)$, $X \in T_x(M)$ and

$$T(c)(Z) = T(c)(X), \qquad T(\widetilde{b})(Z, X) = 0.$$

Now $\widetilde{b}$ is a groupoid morphism over $b\colon M \to Q_V$ and so, taking tangents,

$$
\begin{array}{ccc}
& T(\widetilde{b}) & \\
T(R(c)) & \xrightarrow{\hspace{2cm}} & TG_v \\[4pt]
T(\alpha_V) \Big\downarrow & & \Big\downarrow \; T(\alpha) \\[4pt]
TM & \xrightarrow[\hspace{2cm}]{T(b)} & TQ_V
\end{array}
$$

commutes, where α_V and α are the source projections. Now $T(\alpha_V)(Z, X) = X$ and so it follows that $T(b)(X) = 0$. Similarly, $Z \in T^b M$. It remains to prove that $A(\widetilde{c})(Z) = A(\widetilde{c})(X)$.

Taking $S = R(\widetilde{b})$ with its horizontal structure over $R(c)$, the source and target maps $\widetilde{\beta}_V \colon S \to R(b)$, $(x, y, z, w) \mapsto (w, z)$, and $\widetilde{\alpha}_V \colon (x, y, z, w) \mapsto (y, x)$, are morphisms. Applying the Lie functor, we have $A(\widetilde{\alpha}_V), A(\widetilde{\beta}_V) \colon T^{\widetilde{b}}(R(c)) \to T^b M$ and $A(\widetilde{\alpha}_V)(Z, X) = X$, $A(\widetilde{\beta}_V)(Z, X) = Z$. Now all that is necessary is to observe that on the groupoid level we have $\widetilde{c} \circ \widetilde{\alpha}_V = \widetilde{c} \circ \widetilde{\beta}_V$.

This proves that $T^{\widetilde{b}}(R(c)) \subseteq R(A(\widetilde{c}))$. Since the maps $c, b, \widetilde{c}, \widetilde{b}$ are all submersions, a dimension count shows that equality holds. ∎

This is of course the calculation of $A_H S$ as in [13, §4, p.222]. In this case however the Lie algebroid structure on base $R(c)$ is that of an involutive distribution on $R(c)$ (as well as an action Lie algebroid), and the groupoid structure on base $T^b M$ is the kernel pair of $A(\widetilde{c})$ (as well as an action groupoid). Now we have

$$A^2 = T^{A(\widetilde{c})} T^b M.$$

This is an involutive distribution on $T^b M$ but we prefer to regard it as a submanifold of the double vector bundle $T^2 M$. It is easy to see from the construction, or directly, that $T(p)$ maps A^2 to $T^c M$, and we consequently have a double vector bundle

$$
\begin{array}{ccc}
& T(p) & \\
A^2 & \xrightarrow{\hspace{2cm}} & T^c M \\[4pt]
p_T \Big\downarrow & & \Big\downarrow \; p \\[4pt]
T^b M & \xrightarrow[\hspace{2cm}]{p} & M.
\end{array}
\qquad (27)
$$

Theorem 3.2 *The map $(T(p), p_T) \colon A^2 \to T^c M \oplus T^b M$ is a diffeomorphism.*

The proof of the following lemma is straightforward.

Lemma 3.3 *If $\phi \colon \Omega' \to \Omega$ is a morphism of Lie groupoids over $f \colon M' \to M$, and $\phi^! \colon \Omega' \to f^! \Omega$ is a diffeomorphism, then $A(\phi)^! \colon A\Omega \to f^! A\Omega$ is also a diffeomorphism.*

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PROOF OF 3.2: We know that $R(\widetilde{b}) \to R(c) * R(b)$, $(x, y, z, w) \mapsto ((z, x), (y, x))$, is a diffeomorphism, so $\widetilde{\alpha}_V \colon R(\widetilde{b}) \to R(b)$, $(x, y, z, w) \mapsto (y, x)$, which is a morphism over $R(c) \to M$, $(z, x) \mapsto x$, satisfies the condition of the lemma. Hence

$$T^{\widetilde{b}}(R(c)) \to R(c) * T^b M, \qquad (Z_z, X_x) \mapsto ((z, x), X),$$

is a diffeomorphism. Changing the point of view, it follows that the groupoid morphism $R(A(\widetilde{c})) \to R(c)$, $(Z_z, X_x) \mapsto (z, x)$, over $T^b M \to M$ satisfies the condition of the lemma. Hence $A^2 \to T^b M \oplus T^c M$ is a diffeomorphism. ∎

Before proceeding, we recall some very basic facts about connections [6, XVII§18]. Associated with the double vector bundle $T^2 M$ are the two exact sequences

$$p^! T M \rightarrowtail T^2 M \xrightarrow{T(p)^!} \!\!\!\!\twoheadrightarrow p^! T M, \qquad p^! T M \rightarrowtail T^2 M \xrightarrow{p_T^!} \!\!\!\!\twoheadrightarrow p^! T M$$

where the central terms are respectively the vector bundles $p_T \colon T^2 M \to T M$ and $T(p) \colon T^2 M \to T M$. A connection in M is a map $\mathsf{C} \colon T M \oplus T M \to T^2 M$ which is simultaneously a linear right–inverse for both sequences. We take it that $T(p)(\mathsf{C}(X, Y)) = Y$ and $p_T(\mathsf{C}(X, Y)) = X$. Given a connection C and a vector field X on M, define a vector field X^{C} on $T M$ by $X^{\mathsf{C}}(Y) = \mathsf{C}(Y, X)$. The connection is *flat* if $[X, Y]^{\mathsf{C}} = [X^{\mathsf{C}}, Y^{\mathsf{C}}]$ for all $X, Y \in \mathcal{X}(M)$.

Given $X, Y \in \mathcal{X}(M)$ and $m \in M$, consider $T(Y)(X(m)) - X^{\mathsf{C}}(Y(m))$. This is a vertical tangent vector at $Y(m)$ and so corresponds to an element of $T_m(M)$, which is denoted $\nabla_X(Y)(m)$. This defines the associated *Koszul connection* ∇. There is a bijective correspondence between connections and Koszul connections.

Given a connection C, the *conjugate connection* C' is $\mathsf{C}' = J \circ \mathsf{C} \circ J_0$ where J is the canonical involution in $T^2 M$ and $J_0 \colon T M \oplus T M \to T M \oplus T M$ interchanges the arguments. The corresponding Koszul connection ∇' is given by $\nabla'_X(Y) = \nabla_Y(X) + [X, Y]$.

We now consider the inverse of the diffeomorphism in Theorem 3.2 as constituting a "bipartial" connection in M adapted to the two foliations $R(c)$ and $R(b)$. More precisely, it is a partial connection in the vector bundle $T^b M$ adapted to $R(c)$. The above observations about connections in M apply with the obvious modifications.

Let $\mathcal{X}^c$ and $\mathcal{X}^b$ denote the modules of sections of $T^c M$ and $T^b M$. Given $X \in \mathcal{X}^c$ there is a unique $\overline{X} \in \Gamma_{T^b M}(A^2)$ which projects to X, and this induces as above an operator $\nabla_X^b \colon \mathcal{X}^b \to \mathcal{X}^b$. We thus obtain a "bipartial Koszul connection"

$$\nabla^b \colon \mathcal{X}^c \times \mathcal{X}^b \to \mathcal{X}^b.$$

Since $T(p) \colon A^2 \to T^a M$ is a Lie algebroid morphism, we have $\overline{[X_1, X_2]} = [\overline{X_1}, \overline{X_2}]$ and so A^2 defines a flat bipartial connection. In terms of ∇^b we have $\nabla_{[X_1, X_2]}^b = [\nabla_{X_1}^b, \nabla_{X_2}^b]$ for all $X_1, X_2 \in \mathcal{X}^c$.

Now interchanging c and b, we obtain $A_2 = T^{A(\widetilde{b})} T^c M$ which is a double vector bundle

$$
\begin{array}{ccc}
 & p_T & \\
A_2 & \xrightarrow{\hspace{2cm}} & T^c M \\[4pt]
{\scriptstyle T(p)} \downarrow & & \downarrow\, {\scriptstyle p} \\[4pt]
T^b M & \xrightarrow[\hspace{2cm}]{\;p\;} & M,
\end{array}
$$

and a sub double vector bundle of T^2M. From Theorem 2.2 we now have:

Proposition 3.4 *The canonical involution $J:T^2M \to T^2M$ carries A^2 isomorphically onto A_2.*

A_2 can be considered to be a flat partial connection in T^cM adapted to $R(b)$. In the same way as above we obtain a flat partial Koszul connection $\nabla^c:\mathcal{X}^b \times \mathcal{X}^c \to \mathcal{X}^c$. The following definition now seems reasonable.

Definition 3.5 *Let M be a manifold and let $c:M \to Q_H$ and $b:M \to Q_V$ be two surjective submersions. Then an* infinitesimal affinoid structure *on (M,c,b) is a sub double vector bundle A^2 of T^2M of the form (27) such that $(T(p),p_T):A^2 \to T^cM \oplus T^bM$ is a diffeomorphism and such that $T(p):A^2 \to T^cM$ and $T(p):A_2 = J(A^2) \to T^bM$ are Lie algebroid morphisms.*

Evidently any infinitesimal affinoid structure induces two partial Koszul connections ∇^b and ∇^c as above. By Proposition 3.4 and the definition of conjugate connections, we have (compare Remark 3.2 in [21]):

Proposition 3.6 *In any infinitesimal affinoid structure, ∇^b and ∇^c are conjugate connections; that is, for all $X \in \mathcal{X}^c$, $Y \in \mathcal{X}^b$,*

$$\nabla^c_Y(X) = \nabla^b_X(Y) + [X,Y].$$

Proposition 3.7 *For $X,X_1,X_2 \in \mathcal{X}^c$, $Y,Y_1,Y_2 \in \mathcal{X}^b$,*

$$
\begin{aligned}
\nabla^b_X[Y_1,Y_2] &= [\nabla^b_X(Y_1),Y_2] + [Y_1,\nabla^b_X(Y_2)] + \nabla^b_{\nabla^c_{Y_2}(X)}(Y_1) - \nabla^b_{\nabla^c_{Y_1}(X)}(Y_2), \\
\nabla^c_Y[X_1,X_2] &= [\nabla^c_Y(X_1),X_2] + [X_1,\nabla^c_Y(X_2)] + \nabla^c_{\nabla^b_{X_2}(Y)}(X_1) - \nabla^c_{\nabla^b_{X_1}(Y)}(X_2).
\end{aligned}
$$

PROOF. Since this is a purely formal calculation, we may as well consider an ordinary connection ∇ in M with conjugate connection ∇'. We calculate R', the curvature of ∇'.

By definition, $R'(X,Y)(Z) = \nabla'_{[X,Y]}(Z) - [\nabla'_X,\nabla'_Y](Z)$. Substituting in $\nabla'_X(Y) = \nabla_Y(X) + [X,Y]$, we obtain

$$
\begin{aligned}
R'(X,Y)(Z) &= \nabla_Z[X,Y] - [X,\nabla_Z(Y)] - [\nabla_Z(X),Y] \\
&\quad -\nabla_{[Y,Z]}(X) - \nabla_{[Z,X]}(Y) - \nabla_{\nabla_Z(Y)}(X) + \nabla_{\nabla_Z(X)}(Y).
\end{aligned}
$$

So if $R' = 0$ we get

$$\nabla_Z[X,Y] = [X,\nabla_Z(Y)] + [\nabla_Z(X),Y] + \nabla_{\nabla'_Y(Z)}(X) - \nabla_{\nabla'_X(Z)}(Y).$$

In the case of the bipartial flat connections of an infinitesimal affinoid we can apply this calculation to both ∇^b and ∇^c. ∎

The equations of 3.6 and 3.7 show that T^cM and T^bM are a matched pair of Lie algebroids in the sense of Mokri [17]. The corresponding representations are ∇^c and ∇^b. Indeed (T^cM,T^bM) is the matched pair of Lie algebroids arising from the vacant double groupoid structure on S.

It is now clear that we could equivalently define an infinitesimal affinoid structure on (M,c,b) to be a pair of partial flat connections ∇^c and ∇^b which satisfy the equations of Proposition 3.7.

We now turn to the infinitesimal form of butterfly diagrams, which is easily defined.

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Definition 3.8 *An* infinitesimal butterfly diagram *is a diagram of Lie algebroid morphisms of the form shown in Figure 5, such that the two vertical arrows are action morphisms over*
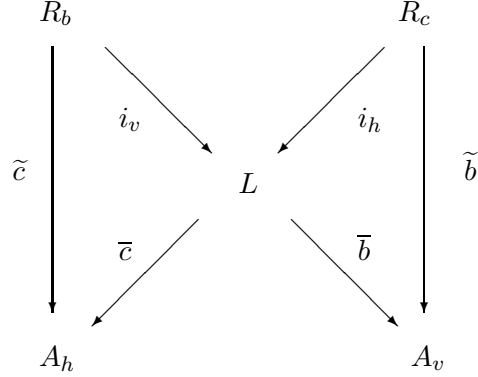

Figure 5:

surjective submersions $c\colon M \to Q_H$ *and* $b\colon M \to Q_V$, *the two upper diagonals are base-preserving embeddings over* c *and* b, *the two lower diagonals are inductors, and the two sequences are exact at their central term.*

Applying the Lie functor to a butterfly diagram clearly leads to an infinitesimal butterfly diagram, since the Lie functor sends action morphisms to action morphisms, inductors to inductors, embeddings to embeddings, and is exact [7].

We now show that an infinitesimal affinoid structure gives rise to an infinitesimal butterfly diagram, provided that the fibres of c and b are simply–connected. The first step holds without any connectivity assumption. Its proof is a straightforward calculation. (A more general result of this type is given in [17].)

Proposition 3.9 *Let* (M, c, b) *have an infinitesimal affinoid structure with associated partial connections* ∇^c *and* ∇^b. *Denote by* L *the vector bundle direct sum* $T^c M \oplus T^b M$. *Then* L *has a Lie algebroid structure over* M *with anchor* $a\colon L \to TM$ *given by* $a(X \oplus Y) = X + Y$, *and bracket*

$$[X \oplus Y, X' \oplus Y'] = \{[X, X'] + \nabla^c_Y(X') - \nabla^c_{Y'}(X)\} \oplus \{[Y, Y'] + \nabla^b_X(Y') - \nabla^b_{X'}(Y)\}$$

for $X, X' \in \mathcal{X}^c$, $Y, Y' \in \mathcal{X}^b$.

Until Theorem 3.10, consider an infinitesimal affinoid structure on (M, c, b) for which the fibres of c and b are simply connected. The partial connections ∇^c and ∇^b can be considered as Lie algebroid morphisms $T^b M \to \mathrm{CDO}(T^c M)$ and $T^c M \to \mathrm{CDO}(T^b M)$ and accordingly integrate to give linear actions of $R(b)$ on the vector bundle $T^c M$ and of $R(c)$ on the vector bundle $T^b M$. Denote these actions by θ^c and θ^b.

We need to show that L quotients over $T^b M$ and $T^c M$ to give Lie algebroids A_v on Q_V and A_h on Q_H. We follow the method of [7, §4]. Say that $X \in \mathcal{X}^c$ is θ^c-*stable* if

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$\theta^c(y,x)X(x) = X(y)$ for all $(y,x) \in R(b)$. This is equivalent to the condition that X be ∇^c-parallel, that is, that $\nabla^c_Y(X) = 0$ for all $Y \in \mathcal{X}^b$. We also say that $X \oplus Y \in \Gamma L$ is θ^c-stable or ∇^c-parallel if X is so.

To show that $T^b M$ is a b-ideal of L we need to verify the following three conditions:

(i). If $X \oplus Y$, $X' \oplus Y'$ are θ^c-stable, then $[X \oplus Y, X' \oplus Y']$ is also;

(ii). If $X \oplus Y \in \Gamma L$ is θ^c-stable, and $Y' \in \mathcal{X}^b$, then $[X \oplus Y, 0 \oplus Y']$ is in $T^b M$;

(iii). The map $L/T^b M \to TM/T^b M$ induced by the anchor of L is equivariant with respect to θ^c and the natural action of $R(b)$ on $TM/T^b M \cong b^! TQ_V$.

The first two conditions are easily checked. For the third, note that the natural action of $R(b)$ on $TM/T^b M$ differentiates to $D_Y(\overline{Z}) = \overline{[Y,Z]}$ where $Y \in \mathcal{X}^b$, $Z \in \mathcal{X}(M)$; here the bar denotes the class modulo $T^b M$. It therefore suffices to check that $\overline{\nabla^c_Y(X)} = \overline{[Y,X]}$ and this follows from the fact that $\nabla^c_Y(X) - [Y,X] = \nabla^b_X(Y) \in \mathcal{X}^b$.

From [7, 4.5] it therefore follows that the vector bundle $T^c M \cong L/T^b M$ descends to a vector bundle A_v on Q_V; that is, $T^c M$ is the vector bundle pullback of A_v over b. Denote the map $T^c M \to A_v$ by $\widetilde{b}$ and the map $L \to L/T^b M \to A_v$ by $\overline{b}$. Further, A_v has a Lie algebroid structure over Q_V with respect to which $\widetilde{b}$ (and therefore $\overline{b}$) is a Lie algebroid morphism. The bracket of two sections $x, x' \in \Gamma A_v$ is obtained by taking the inverse image sections $X, X' \in \Gamma L$; by (i) above, $[X, X']$ is θ^c-stable and therefore descends to a section $[x, x']$ of A_v.

The rank of A_v is the same as that of $T^c M$ and so $\overline{b}$ is an action morphism. Since $\widetilde{b}$ has kernel $T^b M$ by construction, it is an inductor. Carrying out the same construction with $T^c M$ as kernel, we have proved the following.

Theorem 3.10 *Let (M, c, b) have an infinitesimal affinoid structure and assume that c and b have simply connected fibres. Then the above construction yields an infinitesimal butterfly diagram.*

Conversely consider an infinitesimal butterfly diagram as in Figure 5. An inductor of Lie algebroids is essentially a pullback in the category of Lie algebroids, and so its kernel is the vertical bundle of the base map. Thus we have $R_b = T^b M$ and $R_c = T^c M$.

Since $\overline{c} \colon L \to A_h$ is an inductor over c, there is an isomorphism of Lie algebroids

$$L \to TM \underset{c^! TQ_H}{\times} c^! A_h, \qquad Z \mapsto (a_L(Z), \overline{c}^!(Z)),$$

where a_L is the anchor of L. And since $\widetilde{c} \colon T^b M \to A_h$ is an action morphism, we know that $\widetilde{c}^! \colon T^b M \to c^! A_h$ is an isomorphism; denote its inverse by η. Now it is readily checked that

$$L \to T^c M \oplus T^b M, \qquad Z \mapsto (a_L(Z) - \eta(\overline{c}^!(Z))) \oplus \eta(\overline{c}^!(Z))$$

is an isomorphism of vector bundles, and that i_h and i_v are now represented by $X \mapsto X \oplus 0$ and $Y \mapsto 0 \oplus Y$. We can therefore apply the following lemma, whose proof is purely formal.

Lemma 3.11 *Let the vector bundle direct sum $T^c M \oplus T^b M$ have a Lie algebroid structure over M with respect to which $T^c M$ and $T^b M$ are Lie subalgebroids. Define $\nabla^c \colon \mathcal{X}^b \times \mathcal{X}^c \to \mathcal{X}^c$ and $\nabla^b \colon \mathcal{X}^c \times \mathcal{X}^b \to \mathcal{X}^b$ by*

$$[0 \oplus Y, X' \oplus 0] = \nabla^c_Y(X') \oplus -\nabla^b_{X'}(Y).$$

Then ∇^c and ∇^b are flat partial connections and satisfy the relations in Proposition 3.7.

This completes the proof of the following result.

Theorem 3.12 *Let L be an infinitesimal butterfly diagram over $c\colon M \to Q_H$ and $b\colon M \to Q_V$. Then the above construction yields an infinitesimal affinoid structure on M.*

Theorems 3.10 and 3.12 establish an equivalence between infinitesimal affinoid structures and infinitesimal butterfly diagrams when the fibres of c and b are simply connected.

Infinitesimal butterfly diagrams behave in some respects like Morita equivalences: for example, assuming that the fibres of c and b are simply–connected, one can easily prove that A_h is integrable if and only if A_v is integrable. However, given an arbitrary Lie algebroid A, there appears to be no canonical construction of an infinitesimal butterfly diagram with $A_h = A_v = A$, and this indicates severe limitations for the concept.

Lastly in this section consider a generalized principal bundle $P(B, G, p)(Q_V, f)$ [9] as defined in [13, 3.3]. Thus $G \Longrightarrow Q_V$ is a Lie groupoid acting smoothly and freely to the right on a surjective submersion $f\colon P \to Q_V$ with quotient manifold $B = P/G$ and projection $p\colon P \to B$. This defines an affinoid structure on P with respect to $c = p\colon P \to B$ and $b = f\colon P \to Q_V$. The vertical groupoid G_v identifies canonically with the original G. For clarity, denote the elements of G_h by $\langle y, x \rangle$ where $x, y \in P$, $f(x) = f(y)$, and $\langle y, x \rangle = \langle yg, xg \rangle$ for any $g \in G$ with $\beta g = f(x)$.

Since $P(B, G, p)(Q_V, f)$ is presented as a generalization of the concept of principal bundle, it is reasonable to extend the notion of Atiyah sequence to it. (The omitted details in what follows may be found by extending the account in [12, App.A].) Consider the vertical tangent bundle $T^f P$. The action of G on P lifts to a right action of G on $T^f P$ and remains free. Denote elements of $T^f P/G$ by $\langle X_x \rangle$ where $X_x \in T_x^f P$. Then $T^f P/G$ is a vector bundle over B; if $\langle X_x \rangle$ and $\langle Y_y \rangle$ have $p(x) = p(y)$, then there exists $g \in G$ with $y = xg$ and we define $\langle X_x \rangle + \langle Y_y \rangle = \langle Xg + Y \rangle$, as in the standard case. An f-vertical vector field X on P may be defined to be G–*invariant* if $X(xg) = X(x)g$ for all $x \in P$ and $g \in G$ with $f(x) = \beta g$; one obtains a $C(B)$–module of G–invariant vector fields which is in bijective correspondence with the module of sections of $T^f P/G \to B$. Now the bracket of G–invariant vector fields transfers to $\Gamma(T^f P/G)$ and makes it a Lie algebroid with anchor the quotient to $T^f P/G \to TB$ of $T(p)\colon T^f P \to TB$. This might be called the *generalized Atiyah sequence of* $P(B, G, p)(Q_V, f)$. As in the standard case, $T^f P \to T^f P/G$ is an action morphism of Lie algebroids over p.

Proposition 3.13 *The generalized Atiyah sequence just constructed is canonically isomorphic to $A(G_h)$.*

PROOF. The morphism $\widetilde{c}\colon R(f) \to G_h$ over $p\colon P \to B$ is $(y, x) \mapsto \langle y, x \rangle$. It induces a Lie algebroid morphism $A(\widetilde{c})\colon T^f P \to A(G_h)$ which is constant on the orbits of G and therefore induces a Lie algebroid morphism $R\colon T^f P/G \to A(G_h)$ over B. Since $\widetilde{c}$ is an action morphism, it is a fibrewise diffeomorphism, and this property is inherited by $A(\widetilde{c})$ and R. Since R is base–preserving, it is therefore an isomorphism of Lie algebroids. ∎

Note that the method available in the standard case [12, III 3.20] cannot be used here, since P cannot generally be embedded in G_h.

Proposition 3.13 rules out much interest in the notion of generalized Atiyah sequence. Since any Lie groupoid G acts freely to the right on its target projection, yielding a generalized principal bundle whose G_h is again canonically isomorphic to G, any Lie algebroid which is

the Lie algebroid of a Lie groupoid may be constructed as a generalized Atiyah sequence. There is no prospect of extending to generalized Atiyah sequences the very rich theory known for the usual notion of Atiyah sequence, which depends essentially on its transitivity.

Although generalized principal bundles are equivalent to both affinoid structures and butterfly diagrams, it is clear that generalized Atiyah sequences embody only a part of the information in an infinitesimal butterfly diagram.

4   DUALITY   AND   THE   COTANGENT   DOUBLE GROUPOID

As a preliminary to the next section, we show that the cotangent of a double Lie groupoid itself has a double groupoid structure. We begin by recalling the notion of dual for $\mathcal{VB}$-groupoids due to Pradines [19].

Consider a $\mathcal{VB}$-groupoid $(\Omega; G, A; M)$ as in [13, §4], with core $K = \{\xi \in \Omega \mid \widetilde{\alpha}(\xi) = 0_m,\ \widetilde{q}(\xi) = 1_m,\ \exists m \in M\}$. The vector bundle operations on Ω restrict to give K the structure of a vector bundle over M.

Let Ω^* be the dual of Ω as a vector bundle over G. Define a groupoid structure on Ω^* with base K^* as follows. Take $\Phi \in \Omega_g^*$ where $g \in G_m^n$. Then the source and target of Φ in K_m^* and K_n^* respectively are

$$\langle \widetilde{\alpha}_*(\Phi), \kappa \rangle = \langle \Phi, -\widetilde{0}_g \kappa^{-1} \rangle, \quad \kappa \in K_m, \qquad \langle \widetilde{\beta}_*(\Phi), \kappa \rangle = \langle \Phi, \kappa \widetilde{0}_g \rangle, \quad \kappa \in K_n. \qquad (28)$$

For the composition, take $\Psi \in \Omega_h^*$ with $\widetilde{\alpha}_*(\Psi) = \widetilde{\beta}_*(\Phi)$. Any element of Ω_{hg} can be written as a product $\eta\xi$ where $\eta \in \Omega_h$ and $\xi \in \Omega_g$. Now the compatibility condition on Ψ and Φ ensures that

$$\langle \Psi\Phi, \eta\xi \rangle = \langle \Psi, \eta \rangle + \langle \Psi, \xi \rangle \qquad (29)$$

is well defined. The identity element of Ω^* at $\theta \in K_m^*$ is $\widetilde{1}_\theta \in \Omega_{1_m}^*$ defined by

$$\langle \widetilde{1}_\theta, \widetilde{1}_X + \kappa \rangle = \langle \theta, \kappa \rangle, \qquad (30)$$

where any element of Ω_{1_m} can be written as $\widetilde{1}_X + \kappa$ for some $X \in A_m$ and $\kappa \in K_m$.

It is straightforward to check that $(\Omega^*; G, K^*; M)$ is a $\mathcal{VB}$-groupoid, the *dual $\mathcal{VB}$-groupoid to Ω*. The core of Ω^* is the vector bundle $A^* \to M$, with the core element corresponding to $\phi \in A_m^*$ being $\overline{\phi} \in \Omega_{1_m}^*$ defined by

$$\langle \overline{\phi}, \widetilde{1}_X + \kappa \rangle = \langle \phi, X + \partial_A(\kappa) \rangle \qquad (31)$$

for $X \in A_m$, $\kappa \in K_m$. Here $\partial_A \colon K \to A$ is the map defined by $\widetilde{\beta} \colon \Omega \to A$ as in [13, §5].

Now consider a morphism of $\mathcal{VB}$-groupoids which preserves the lower groupoids:

$$(F; \mathrm{id}_G, f; \mathrm{id}_M) \colon (\Omega; G, A, M) \to (\Omega'; G, A'; M)$$

and denote the core morphism $K \to K'$ by f_K. The proof of the following result is simple.

Proposition 4.1 *The dual morphism $F^* \colon \Omega'^* \to \Omega^*$ is a morphism of the dual $\mathcal{VB}$-groupoids, with base map $f_K^* \colon K'^* \to K^*$ and core morphism $f^* \colon A'^* \to A^*$.*

For any Lie groupoid $G \rightrightarrows M$, the dual of the tangent $\mathcal{VB}$-groupoid $(TG; G, TM; M)$ is the cotangent groupoid $(T^*G; G, A^*G; M)$; the conventions used above give precisely the structure given in [15, §7]. If now $G \rightrightarrows M$ is a Poisson groupoid, $\pi_G^\#: T^*G \to TG$ is a morphism of $\mathcal{VB}$-groupoids with respect to the base maps id_G and $a_*: A^*G \to TM$ (see [1] or [15, 8.1]). It now follows from 4.1 and from the skewsymmetry of $\pi_G^\#$, that the core map of $\pi_G^\#$ is $-a_*^*$.

The duality for $\mathcal{VB}$-groupoids can be extended to a duality between $\mathcal{LA}$-groupoids and what one might call $\mathcal{PVB}$–groupoids: $\mathcal{VB}$-groupoids $(\Omega; G, E; M)$ in which Ω and E carry Poisson structures making $\Omega \rightrightarrows E$ a Poisson groupoid. This notion will be developed elsewhere.

We now give a double version of the cotangent groupoid. Consider a double Lie groupoid $(S; H, V; M)$ with core groupoid $C \rightrightarrows M$ as in [2] or [13, §2, p.197]. Denote by $A_H S$ and $A_V S$ the Lie algebroids of the two groupoid structures on S.

The standard cotangent groupoid structure ([3] or [19]) gives T^*S groupoid structures over both $A_H^* S$ and $A_V^* S$. In turn, since both $A_H S$ and $A_V S$ are $\mathcal{LA}$-groupoids with core AC, as in 1.6, the duals $A_H^* S$ and $A_V^* S$ both have groupoid structures on base A^*C.

Theorem 4.2 *With the structures just described, T^*S is a double Lie groupoid*

$$
\begin{array}{ccc}
 & \widetilde{\alpha}_{*H}, \widetilde{\beta}_{*H} & \\
T^*S & \Longrightarrow & A_H^* S \\[2pt]
\widetilde{\alpha}_{*V}, \widetilde{\beta}_{*V} \; \Big\Downarrow & & \Big\Downarrow \; \alpha_{*V}, \beta_{*V} \\[2pt]
A_V^* S & \Longrightarrow & A^*C \\
 & \alpha_{*H}, \beta_{*H} &
\end{array}
\tag{32}
$$

PROOF. Using the groupoid structures defined in (28)—(30) on the side groupoids and, on T^*S, the ordinary cotangent groupoid structures which are a special case of these, this is a long but straightforward verification. ∎

Theorem 4.3 *The core groupoid of (32) is naturally isomorphic with the cotangent groupoid $T^*C \rightrightarrows A^*C$ of the core of S.*

PROOF. Define a map $E: T_C S \to TC$ which sends each vector tangent to S at a point of the core, to a vector at the same point tangent to the core itself. Take $c \in C$ with $v = \partial_V(c)$, $h = \partial_H(c)$, where $v \in V_m$, $h \in H_m$. Represent an element $\xi \in T_c S$ as

$$
\begin{array}{ccc}
w & W & x \\
\hline
Z & \;\xi\; & X \\
\hline
z & Y & y
\end{array}
$$

where $X = T(\widetilde{\alpha}_H)(\xi) \in T_{1_m^V} V$, $x = T(\beta_V)(X) = T(\alpha_H)(W) \in T_m M$, et cetera, and define

$$
E(\xi) = \xi - T(L_c^V)T(\widetilde{1}^H)(X - T(1^V)(x)) - T(L_c^H)T(\widetilde{1}^V)(Y - T(1^H)(z)),
\tag{33}
$$

29

where L_c^V and L_c^H denote left translation in the two groupoid structures on S.

Take $\sigma \in T_c^*C$ and define $\Sigma \in T_c^*S$ by $\Sigma(\xi) = \sigma(E(\xi))$, $\xi \in T_cS$. We must show that Σ is a core element. Denote $\widetilde{\alpha}_{*V}(\Sigma) \in A_V^*S|_{1_m^H}$ by ρ, and $\alpha_{*H}(\rho) \in A_m^*C$ by θ. Then for all $\kappa \in A_mC$,

$$\langle \theta, \kappa \rangle = -\rho(T(\widetilde{\iota}_H)(\kappa)) = \Sigma(T(L_c^V)T(\widetilde{\iota}_V)T(\widetilde{\iota}_H)(\kappa))$$

where $\widetilde{\iota}_H$ and $\widetilde{\iota}_V$ are the two inversions in S.

We first prove that $\rho = 1_\theta^{*V}$. Take any $\xi = A(\widetilde{1}^H)(X) + \kappa \in A_VS|_{1_m^H}$ where $X \in A_mV$, $\kappa \in A_mC$. Then $1_\theta^{*V}(\xi) = \langle \theta, \kappa \rangle$ and $\rho(\xi) = -\Sigma(T(L_c^V)T(\widetilde{\iota}_V)(\xi))$. Now although it is natural to consider κ as an element of A_VS in this context, it may also be regarded as an element of A_HS and we then get

$$T(\widetilde{\iota}_H)(\kappa) = T(\widetilde{1}^H)(Z) - \kappa$$

where $Z = \partial_{AV}(\kappa)$. Next, applying E to $T(L_c^V)T(\widetilde{\iota}_V)T(\widetilde{1}^H)(Z)$, or equally with X in place of Z, gives $0 \in T_cC$, where we use $\widetilde{\iota}_V \circ \widetilde{1}^H = \widetilde{1}^H \circ \iota_V$. Putting these together, the equation for ρ follows.

We must also prove that $\theta \in A_m^*C$ is the source of σ with respect to $T^*C \Longrightarrow A^*C$. For $\kappa \in A_mC$ the above shows that

$$\langle \theta, \kappa \rangle = -\Sigma(T(L_c^V)T(\widetilde{\iota}_V)(\kappa)).$$

Calculating E of the argument on the right hand side, we obtain

$$T(L_c^V)T(\widetilde{\iota}_V)(\kappa) + T(L_c^H)T(\widetilde{1}^V)(T(1^H)(a_HW) - W), \tag{34}$$

where $W = \partial_{AH}(\kappa)$. Write $\kappa = \frac{d}{dt}c_t\big|_0$ where the curve c_t in C is of the form

Then the first term in (34) is $\frac{d}{dt}c \boxminus c_t^{-V}\big|_0$ and the second is $\frac{d}{dt}c \boxdot \widetilde{1}_{h_t^{-1}}^V\big|_0$. The sum is therefore the derivative of the product shown in Figure 6, and in terms of the multiplication $\boxtimes$ and the inversion $c \mapsto c^{-C}$ in $C \Longrightarrow M$, this product is $c \boxtimes c_t^{-C}$ (see [13, §2, p.197] or [2]). The derivative is therefore $T(L_c^C)T(\iota_C)(\kappa)$ and so we have $\langle \theta, \kappa \rangle = \langle \alpha_{*C}(\sigma), \kappa \rangle$.

The proof that $\widetilde{\alpha}_{*H}(\Sigma) = 1_\theta^{*H}$ now follows in the same way. One likewise checks that $\beta_{*C}(\sigma) = \beta_{*H}(\widetilde{\beta}_{*V}(\Sigma))$ and that the multiplications correspond.

This proves that $\sigma \mapsto \Sigma$ is an isomorphism into the core of T^*S. We leave the reader to check that the image is the whole of T^*C. ∎

In point of fact, T^*S is a triple structure, each of its four spaces being also a vector bundle over the corresponding space of $(S; H, V; M)$, and all the groupoid structure maps being vector bundle morphisms.
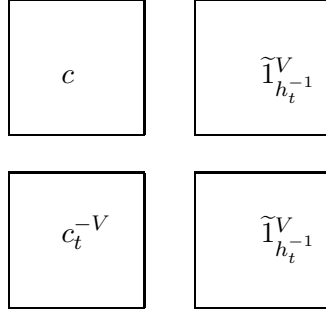
Figure 6:

5 SYMPLECTIC DOUBLE GROUPOIDS AND POISSON DOUBLE GROUPOIDS

In introducing the concept of Poisson groupoid in [20], Weinstein defined two Poisson groupoids on the same base manifold to be *dual* if the Lie algebroid dual of each is isomorphic to the Lie algebroid of the other, and described a programme for showing that, at least locally, Poisson groupoids in duality arise as the bases of a symplectic double groupoid. In [10] he and Lu carried out this programme globally for Poisson groups. However neither paper fully describes the general process by which a symplectic double groupoid gives rise to a pair of Poisson groupoids in duality.

In this section we use the apparatus of double Lie algebroids to show that the side groupoids of a symplectic double groupoid are Poisson groupoids whose Lie bialgebroids [15] are isomorphic under a duality, and that the core of the double groupoid provides a symplectic realization of the double base. The double Lie algebroid structure also provides symplectic realizations of the linearized Poisson structures. The proof is conceptual and reasonably simple. The key is the notion of core as developed in [2] and [13, §2].

In view of the integrability of the double base of a symplectic double groupoid, it seems likely that there exist pairs of Poisson groupoids in duality which do not integrate globally to a symplectic double groupoid.

As with ordinary groupoids, it is actually easier to study symplectic structures by specializing from the Poisson case. The notion of Poisson double groupoid defined here includes (see [13, §3]) the Poisson affinoids of [5].

In what follows we will repeatedly use the following simple result.

Proposition 5.1   (i). *Let $(\phi; \phi_H, \phi_V; \phi_M): (S; H, V; M) \to (S'; H', V'; M')$ be a morphism of double groupoids with core morphism $\phi_C: C \to C'$. Then $\partial'_H \circ \phi_C = \phi_H \circ \partial_H$ and $\partial'_V \circ \phi_C = \phi_V \circ \partial_V$.*

(ii). *Let $(\phi; \phi_G, \phi_A; \phi_M): (\Omega; G, A; M) \to (\Omega'; G', A'; M')$ be a morphism of $\mathcal{L}\mathcal{A}$-groupoids with core Lie algebroid morphism $\phi_K: K \to K'$. Then $\partial'_{AG} \circ \phi_K = A(\phi_G) \circ \partial_{AG}$ and $\partial'_A \circ \phi_K = \phi_A \circ \partial_A$.*

Definition 5.2 *A* Poisson double groupoid *is a double Lie groupoid $(S; H, V; M)$ together with a Poisson structure on S with respect to which both groupoid structures on S are Poisson groupoids.*

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For the theory of ordinary Poisson groupoids, see [20], [23], [15]. For a Poisson groupoid $G \Longrightarrow P$ we take the Poisson structure on the base to be $\pi_P^\# = a_* \circ a^* = -a \circ a_*^*$. This convention is opposite to that used in [15], but is necessary in order that the two core morphisms $a^*: T^*P \to A^*G$ and $-a_*^*: T^*P \to AG$ of the cotangent $\mathcal{LA}$-groupoid be Lie algebroid morphisms: see the remark following 4.1.

The two Poisson groupoid structures on S induce maps $\widetilde{a}_{*H}: A_V^*S \to TH$ and $\widetilde{a}_{*V}: A_H^*S \to TV$, the anchors for the Lie algebroid structures on the duals of $A_VS \to H$ and $A_HS \to V$, with respect to which $\pi_S^\#: T^*S \to TS$ is a morphism of each of the two groupoid structures on T^*S. From the following result it follows that $\pi_S^\#$ is actually a morphism of double groupoids over a map $a_{*C}: A^*C \to TM$.

Lemma 5.3 Let $(S; H, V; M)$ and $(S'; H', V'; M')$ be double Lie groupoids and let $\phi: S \to S'$, $\phi_H: H \to H'$ and $\phi_V: V \to V'$ be maps such that (ϕ, ϕ_H) and (ϕ, ϕ_V) are morphisms of the two ordinary groupoid structures on S and S'. Then there is a unique map $\phi_M: M \to M'$ such that $(\phi; \phi_H, \phi_V; \phi_M)$ is a morphism of double groupoids.

PROOF. Take $m \in M$. The double identity 1_m^2 can be written both as $\widetilde{1}_{1_m^H}^V$ and as $\widetilde{1}_{1_m^V}^H$. Its image under ϕ is therefore an identity for both top structures on S', and must therefore be a double identity $1_{\phi_M(m)}^2$. It also follows that ϕ_H, ϕ_V and ϕ_M commute with the source and target projections.

Now since $\phi(\widetilde{1}_h^V) = \widetilde{1}_{\phi_H(h)}^V$ for all $h \in H$ and $\widetilde{1}_{h_1}^V \boxempty \widetilde{1}_{h_2}^V = \widetilde{1}_{h_1 h_2}^V$ for all compatible $h_1, h_2 \in H$, it follows that ϕ_H is a morphism over ϕ_M. Similarly for ϕ_V. ∎

Returning to the Poisson double groupoid S, the bases H and V acquire Poisson structures which we take to be $\pi_H^\# = \widetilde{a}_{*H} \circ \widetilde{a}_V^*$ and $\pi_V^\# = \widetilde{a}_{*V} \circ \widetilde{a}_H^*$. We will prove below that H and V are Poisson groupoids with respect to these structures.

First note that the core morphism $T^*C \to TC$ of $\pi_S^\#$ defines a Poisson structure on C; denote this by $\pi_C^\#$. The core is a morphism of groupoids over $a_{*C}: A^*C \to TM$ and $C \Longrightarrow M$ is therefore a Poisson groupoid. Give M the Poisson structure $\pi_M^\# = a_{*C} \circ a_C^*$ induced from C.

Since $S_V \Longrightarrow H$ is a Poisson groupoid, its Lie algebroid dual A_V^*S has a Lie algebroid structure with anchor $\widetilde{a}_{*H}$. Similarly A^*C has a Lie algebroid structure with anchor a_{*C}.

Proposition 5.4 With respect to these structures $(A_V^*S; H, A^*C; M)$ is an $\mathcal{LA}$-groupoid.

PROOF. It must be proved that α_{*H} and β_{*H} and the groupoid multiplication are Lie algebroid morphisms. This proceeds as in the case of the cotangent $\mathcal{LA}$-groupoid of a Poisson groupoid: to prove β_{*H} a Lie algebroid morphism it suffices ([15, 6.1] or [8, 6.6]) to prove that the dual map

$$AC * H \to A_VS, \qquad (\kappa, h) \mapsto \kappa \widetilde{0}_h,$$

is Poisson with respect to the linearized structures on AC and A_VS. ∎

We may therefore consider both T^*S and TS to be triple structures as in Figure 7: precisely, they are double groupoid objects in the category of Lie algebroids. We will call such structures $\mathcal{LA}$–double groupoids. The three double structures which involve T^*S or TS we call the *upper faces* or *upper structures*, the other three being the *lower faces* or *lower*

$$
\begin{array}{ccc}
T^*S & \Longrightarrow & A_H^*S \\
\Downarrow & & \Downarrow \\
A_V^*S & \Longrightarrow & A^*C \\
& S & \Longrightarrow & V \\
& \Downarrow & & \Downarrow \\
& H & \Longrightarrow & M
\end{array}
\qquad
\begin{array}{ccc}
TS & \Longrightarrow & TV \\
\Downarrow & & \Downarrow \\
TH & \Longrightarrow & TM \\
& S & \Longrightarrow & V \\
& \Downarrow & & \Downarrow \\
& H & \Longrightarrow & M
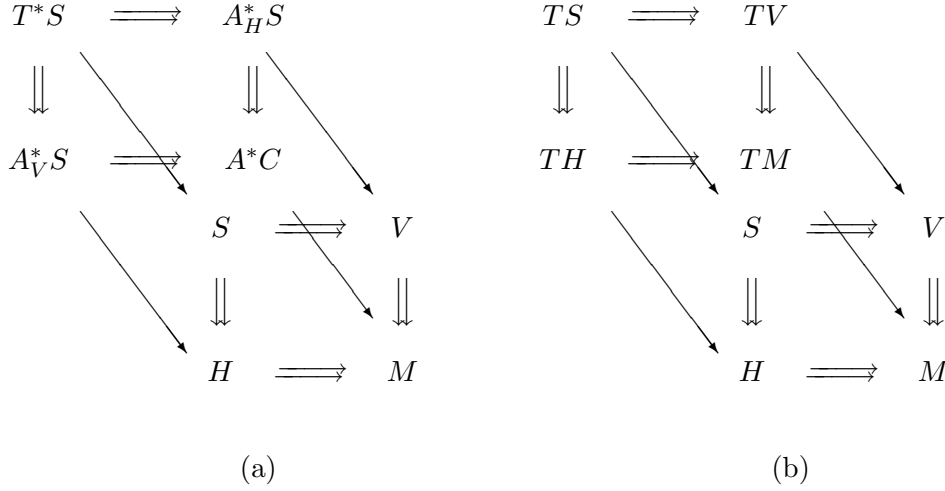\end{array}
$$

(a)   (b)

Figure 7:

structures. That $\pi_S^{\#}$ respects all twelve ordinary structures follows from the hypothesis and basic results for ordinary Poisson groupoids.

In what follows we will use the following result repeatedly. The proof is straightforward.

Proposition 5.5   (i). *The core of each upper face in an $\mathcal{LA}$-double groupoid is a Lie groupoid or Lie algebroid over the core of the opposite lower face, and forms an $\mathcal{LA}$-groupoid with respect to the structures of the bases of the two cores.*

   (ii). *The restriction of a morphism of $\mathcal{LA}$-double groupoids to the core of an upper face is a morphism of $\mathcal{LA}$-groupoids.*

The map $\widetilde{a}_{*H}\colon A_V^*S \to TH$ is a morphism of the $\mathcal{LA}$-groupoids which form the bottom faces in Figure 7, the other maps being $a_{*C}\colon A^*C \to TM$ and id_H. Its core map $A^*V \to AH$ is therefore a morphism of Lie algebroids: denote it by D_H.

By [13, §5], the anchor $\widetilde{a}_V\colon A_VS \to TH$ is a morphism of $\mathcal{LA}$-groupoids over $a_V\colon AV \to TM$ and id_H, with core morphism $\partial_{AH}\colon AC \to AH$. Its dual $\widetilde{a}_V^*\colon T^*H \to A_V^*S$ is therefore a morphism over $\partial_{AH}^*\colon A^*H \to A^*C$ and id_H, with core map $a_V^*\colon T^*M \to A^*V$. It follows that $\pi_H^{\#} = \widetilde{a}_{*H} \circ \widetilde{a}_V^*$ is a morphism of groupoids over $a_{*C} \circ \partial_{AH}^*\colon A^*H \to TM$; since it is also a morphism of Lie algebroids over H, it is a morphism of $\mathcal{LA}$-groupoids, and the core morphism is $D_H \circ a_V^*\colon T^*M \to AH$. This proves the first part of the following theorem.

Theorem 5.6 *With the induced structures, $H \Longrightarrow M$ is a Poisson groupoid with*

$$
a_{*H} = a_{*C} \circ \partial_{AH}^* = -a_V \circ D_H^*.
$$

The induced Poisson structure on M coincides with that induced by C.

For the last statement, recall from [13, §5, p.230] that $a_C = a_H \circ \partial_{AH}$. Hence $a_{*C} \circ a_C^* = a_{*C} \circ \partial_{AH}^* \circ a_H^* = a_{*H} \circ a_H^*$. The second equation for a_{*H} follows by noting that the core map

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for $\pi_H^\#$ is $D_H \circ a_V^*$ and the negative dual of this is equal to the base map, by the remark following 4.1.

The same process can be carried out with H and V interchanged. We now have two Lie algebroid morphisms, $D_H : A^*V \to AH$ and $D_V : A^*H \to AV$, defined as the cores of $\widetilde{a}_{*H}$ and $\widetilde{a}_{*V}$ respectively.

Theorem 5.7 $D_V^* = -D_H$.

PROOF. We can regard $\widetilde{a}_{*V}$ as the base map for the cotangent $\mathcal{LA}$-groupoids which form the top faces of Figure 7. From the remark following 4.1 it follows that the core map for these faces is $-\widetilde{a}_{*V}^* : T^*V \to A_H S$. In turn, as in 5.5, the cores of the top faces of Figure 7(a) are groupoids over the cores of the bottom faces. The base map $A^*V \to AH$ for $-\widetilde{a}_{*V}^*$ is, by 4.1, the negative dual of the core map of $\widetilde{a}_{*V}$; that is, it is $-D_V^*$.

By the commutativity properties of the triple structures, one can see that the base map of the core map of the top faces is the same as the core map for the bottom faces. But the core map for the bottom faces is D_H. ∎

We now specialize to symplectic double groupoids, as considered in [3], [20] and [10]. We take the signs in each structure to be the same.

Definition 5.8 *A* symplectic double groupoid *is a double Lie groupoid* $(S; H, V; M)$ *together with a symplectic structure on* S *such that both groupoid structures on* S *are symplectic groupoids.*

Since $\pi_S^\# : T^*S \to TS$ is an isomorphism of double groupoids, it follows that the base maps $\widetilde{a}_{*H} : A_V^* S \to TH$ and $\widetilde{a}_{*V} : A_H^* S \to TV$ are $\mathcal{LA}$-groupoid isomorphisms, and hence their core maps $D_H : A^*V \to AH$ and $D_V : A^*H \to AV$ are Lie algebroid isomorphisms. Further, the core map $\pi_C^\# : T^*C \to TC$ of $\pi_S^\#$ is an isomorphism, and so $C \Longrightarrow M$ is a symplectic groupoid. This proves the following result.

Theorem 5.9 *Let* $(S; H, V; M)$ *be a symplectic double groupoid with core* $C \Longrightarrow M$. *Then* C *is a symplectic groupoid realizing* M *and the Lie bialgebroids* (AH, A^*H) *and* (A^*V, AV) *of the side Poisson groupoids are canonically isomorphic.*

The negative dual of $\widetilde{a}_{*H}$ gives an isomorphism of $\mathcal{LA}$-groupoids from $(T^*H; A^*H, H; M)$ to $A_V S$, the side maps being D_V and id_H. It follows that $A_V S \Longrightarrow AV$ may be identified with the symplectic groupoid $T^*H \Longrightarrow A^*H$, thus giving a symplectic realization of the linearized Poisson structure on AV. Further, the Lie algebroid structure of $A^2 S \to AV$ is $AT^*H \cong T^*(A^*H) \cong T^*(AV)$. Similarly the vertical structure $A^2 S \to AH$ is $T^*(AH) \to AH$.

It is worthwhile reexamining Examples 2.13 and 1.18 in the light of this section.

Finally, T^*S is a symplectic double groupoid for any double Lie groupoid S, and so we have the following result.

Theorem 5.10 *Let* $(S; H, V; M)$ *be a double Lie groupoid. Then* $A_V^* S \Longrightarrow A^*C$ *and* $A_H^* S \Longrightarrow A^*C$ *are Poisson groupoids in duality.*

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